Deliberation Monotonic Social Choice*

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Abstract

Deliberation is often said to be important in a democratic society, but it is not known which voting rules can appropriately reflect the consequences of deliberation. We introduce some axioms that capture this point and identify voting rules that satisfy our axioms in a one-dimensional spatial model. We first show that a voting rule satisfies *individual deliberation monotonicity* and other standard axioms if and only if it is either the leftest rule or the rightest rule. We then characterize a class of voting rules that satisfy *efficiency*, *anonymity*, *neutrality*, *strategy-proofness*, and *total deliberation monotonicity*. We also obtain counterparts of these characterizations in a binary choice model.

Keywords: Deliberation monotonicity, Deliberative democracy, Median rule, Unanimity rule

JEL Codes: D71, D63.

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1 Introduction

Deliberation has played an important role in theory and practice of democracy since ancient Greece.[1] James Fishkin (2009), a leading researcher on deliberative democracy, points out that deliberation is necessary to exercise democratic decisions in a meaningful way.[2] Also, Kenneth Arrow (1951) says that voting is an essential method to make political decisions in the same way that the market mechanism is essential to make economic decisions. Voting and deliberation are two of the most important methods of social choice in a democracy. However, there has not been any discussion about the types of voting rules that can appropriately reflect the consequences of deliberation.

It is widely considered and observed that voters' preferences change through deliberation (e.g., Manin, Stein and Mansbridge 1987; Miller 1992; Goeree and Yarive 2011; List, Luskin, Fishkin and McLean 2013). The changes in voters' preferences cause a change in voting outcomes; however, it is not obvious that a voting rule appropriately reflects the consequences of deliberation. For example, consider the reaction of the plurality rule when preferences change. Suppose that there are seven voters whose preferences are single-peaked on eight alternatives $x_1, x_2, \ldots, x_8$ with $x_1 < x_2 < \cdots < x_8$. Let their peaks be arranged as

$$p_1 = p_2 = x_1, \quad p_3 = p_4 = p_5 = x_2, \quad p_6 = p_7 = x_8.$$

Then x_2 is the plurality winner with three votes (Figure 1). Now suppose that a group of voters $S = \{3, 4, 5, 6, 7\}$ deliberate and their peaks become more similar:

$$p'_3 = x_3, \quad p'_4 = x_4, \quad p'_5 = x_5, \quad p'_6 = x_6, \quad p'_7 = x_7.$$

[1]Athenian democracy, where Athenians discussed political issues before voting, is one of the origins of deliberative democracy. More recently, political philosophers and theorists of democracy, for example, Marquis de Condorcet, John Stuart Mill, Jürgen Habermas, and James Fishkin emphasize the importance of deliberation. Some theorists respect normative aspects of deliberation but, in effect, we treat deliberation as a preference transformation process in this paper.

[2]For example, information sharing and consensus making in well-organized deliberation help us avoid serious failures of democracy such as "rational ignorance" or "tyranny of the majority."

Then the plurality winner moves to x_1 (Figure 2). The direction of the change in social choice is opposite from the direction of the changes in voters' preferences. In fact, the winner chosen before deliberation (x_2) is closer to the deliberation group's peak points than the winner chosen after deliberation (x_1). Though deliberation made voters' preferences more informed, the plurality rule failed to reflect it. In this sense, the plurality rule cannot appropriately reflect the consequence of deliberation. Though simple, this example shows that appropriately reflecting the changes in preferences is non-trivial.

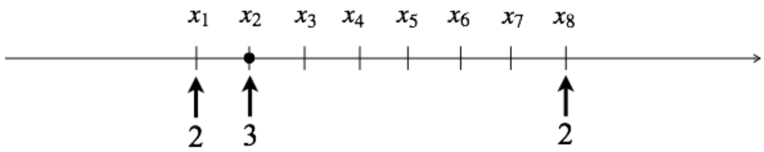

Figure 1: Before deliberation, x_2 is the plurality winner.

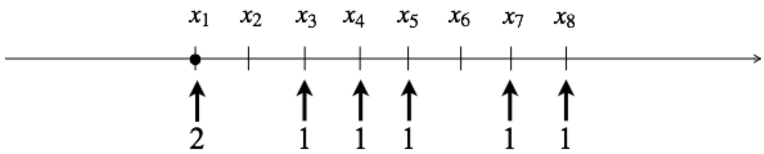

Figure 2: After deliberation, x_1 is the plurality winner.

In short, if the direction of the change in social choice is converse to the direction of the changes in voters' preferences, then voters who deliberated may regret the deliberation. This argument suggests the importance of designing a voting rule that is *monotonic* with respect to the change in preferences. In this paper, we formalize monotonicity axioms with respect to deliberation and search for voting rules that satisfy these axioms in a one-dimensional spatial model with Euclidean preferences (Black 1948a,b; Downs 1957).

Our main axiom is *deliberation monotonicity.* It requires that a voting rule does not increase the *distance* between a preference profile of a deliberation group and a social outcome with respect to *after-deliberation preferences.* If a rule satisfies this property, the social choice after deliberation is more acceptable to voters than that obtained before deliberation. We also presents variants of deliberation monotonicity. We show that the class of threshold rules, including the median rules, satisfies deliberation monotonicity. Moreover, we characterize the rules that satisfy a set of desirable properties, including deliberation monotonicity. We first show that a voting rule satisfies efficiency, anonymity, neutrality, and *individual deliberation monotonicity* if and only if it is either the leftest rule or the rightest rule. We then show that a voting rule satisfies efficiency, anonymity, neutrality, strategy-proofness, and *total deliberation monotonicity* if and only if it is either one of the leftest rule, rightest rule, left median rule, and the right median rule.

One of the pioneers on the mathematical analysis of searching for desirable voting rules is Marquis de Condorcet (1785). Condorcet (1785) also presents the *jury model*, where jurors are faced with choosing one of two alternatives—typically convict or acquit—and wanting to make a correct collective decision, but each juror does not know which alternative is correct. Many recent theoretical studies on impacts of deliberation have used this jury model framework (e.g., Couglan 2000; Austen-Smith and Fedderson 2005, 2006; Gerardi and Yariv 2007; Jackson and Tan 2013). Our research is different from these preceding studies in two ways. First, previous studies conduct game-theoretic analyses focusing on some voting rules. In contrast, we impose some assumptions on how preferences change and axiomatically analyze the properties of voting rules. That is, we do not examine game-theoretic aspects of deliberation, but assumptions on how preferences change are based on results of previous studies.[3] Second, most studies on deliberative committee decision-making are examined in a binary choice model, but we mainly conduct our analysis in a spatial model. This setting applies to many situations. A prominent example is deliberative monetary valuation on an environmental asset (e.g., Spash 2007). However, our main results do not depend on the richness of the alternative set. Indeed, we obtain counterparts of the main results using a binary choice model.

Experimental research on deliberation is also studied. Fishkin (1991, 2009) conducts social experiments of *deliberative public opinion-polls* to attempt to rec-

[3]We do not assume that voters try to manipulate information and voting outcomes in the same way as Condorcet.

oncile deliberation with democracy. List, Luskin, Fishkin and McLean (2013) present empirical tests using data from deliberative public-opinion polls. Guarnashelli, McKelvey and Palfrey (2000), and Goeree and Yarive (2011) study deliberative jury decisions in labs.[4] Some empirical results suggest that deliberation produces substantive agreement. Therefore, we concentrate on certain types of preference transformations that make voters' preferences more similar.

This paper proceeds as follows. In Section 2, we introduce our model and definitions. In Section 3, we present the main results in a spatial model. In Section 4, we present counterparts of the main results in a binary choice model. In Section 5, we deal with the general class of distance functions. Section 6 concludes discussion. Some proofs are collected in the appendices.

2 Definitions

2.1 Model

Let $I = \{1, 2, \ldots, n\}$ be the set of voters with $n \geq 3$. The set of alternatives is the unit interval $X = [0, 1]$. Later, we examine a binary choice model with $X = \{0, 1\}$. Each $i \in I$ has a Euclidean preference on X that is characterized by his peak point $p_i \in X$, that is, for each $x, y \in X$, i weakly prefers x to y if and only if $|p_i - x| \leq |p_i - y|$. Because each voter's preference is a Euclidean ordering, we can identify his preference with his peak point. We denote a peak profile by $p \equiv (p_1, p_2, \ldots, p_n) \in X^I$ and a peak profile of the voters in $S \subset I$ by $p_S \equiv (p_i)_{i \in S} \in X^S$. For each integer k, we often write $p[k]$ for the k-th lowest value among $p_1, p_2, \ldots, p_n$. For example, if $p = (0, \frac{1}{2}, 1, \frac{1}{2})$, $p[1] = 0, p[2] = \frac{1}{2}, p[3] = \frac{1}{2}$, and $p[4] = 1$.

A *voting rule*, or simply a *rule*, is a function $f : X^I \to X$ that maps each peak profile $p \in X^I$ to a collective outcome $f(p) \in X$. A *median rule* f^M is a rule that maps each $p \in X^I$ to a median among $p_1, p_2, \ldots, p_n$, that is, $f^M(p) \in$

[4] Goeree and Yarive (2011) report that deliberation diminishes differences of institutional design and uniformly improves efficiency. List, Luskin, Fishkin and McLean (2013) show that deliberation increases proximity to single-peakedness.

$\{p_1, p_2, \ldots, p_n\}$ satisfies

$$|\{i \in I : p_i \leq f^M(p)\}| \geq \frac{n}{2} \quad \text{and} \quad |\{i \in I : p_i \geq f^M(p)\}| \geq \frac{n}{2}.$$

There are at most two medians if n is even. The median rule f that always chooses the left median is called the *left median rule*, f^{LM}. The median rule that always chooses the right median is called the *right median rule*, f^{RM}. If n is odd, $f^{LM} = f^{RM}$.

The next class of rules is often discussed in our main sections. For each $k = 1, 2, \ldots, n$, the k-th rule f^k is the rule such that for each $p \in X^I$,

$$f^k(p) \equiv p[k].$$

Several rules belong to the class of k-th rules. For example, the median rule is the $\frac{n+1}{2}$-th rule when n is odd. The *leftest rule* f^L is also one of the k-th rules defined as for each $p \in X^I$,

$$f^L(p) \equiv p[1].$$

Similarly, the *rightest rule* f^R is defined as for each $p \in X^I$,

$$f^R(p) \equiv p[n].$$

Next, we introduce a function that measures the *distance* between preferences of a group and an outcome. A *distance function*

$$d : (\bigcup_{S \subset I} X^S) \times X \to \mathbb{R}_+$$

is a function such that for each $x, y \in X$, each profile $p \in X^I$, each voter $i \in I$, and each group $S \subset I$,

- $d(p_i, x) = 0$ if and only if $p_i = x$,
- $d(p_i, x) \leq d(p_i, y)$ if and only if i weakly prefers x to y, (1)
- $d(p_S, x) \leq d(p_S, y)$ if j weakly prefers x to y for all $j \in S$.

For each $S \subset I$ and each outcome $f(p) \in X$, $d(p_S, f(p))$ indicates the *distance* between group S and outcome $f(p)$. We can interpret $d(p_S, f(p))$ as *dissatisfaction* of group S with outcome $f(p)$.

We consider a situation where a group of voters do a deliberation before they vote. By the deliberation of a group of voters, their preferences change. Namely, if a group $S \subset I$ do a deliberation, its preference profile $p_S \in X^S$ is transformed into $p'_S \in X^S$. We often write $p' \in X^I$ for an *after-deliberation preference profile*, that is, $p' = (p'_S, p_{-S})$, where S is a deliberation group. We regard deliberation as an information sharing and consensus making process. Therefore, in our analysis, we focus on certain types of deliberation that make voters' preferences more similar.[5] We present two plausible properties of preference transformations by deliberation. (i) For preference profiles $p, p' \in X^I$ and a group $S \subset I$, call p' a *weakly centering transformation* of p at S if for all $i \in S$,

$$\min_{j \in S} p_j \leq p'_i \leq \max_{j \in S} p_j,$$

and for all $i \notin S$, $p'_i = p_i$. Denote by $W(p, S)$ the set of all weakly centering transformations of p at S. (ii) For preference profiles $p, p' \in X^I$ and a group $S \subset I$, call p' a *centering transformation* of p at S if for all $i \in S \setminus \{m\}$,

$$\begin{aligned} p_i \leq p_m &\Longrightarrow p_i \leq p'_i \leq p'_m, \\ p_m \leq p_i &\Longrightarrow p'_m \leq p'_i \leq p_i, \end{aligned}$$

where $m \in S$ is a median voter among p_S, and for all $i \notin S$, $p'_i = p_i$.[6] Denote by $C(p, S)$ the set of all centering transformations of p at S. Clearly, $C(p, S) \subset W(p, S)$ for each $p \in X^I$ and $S \subset I$. For example, let

$$p \equiv (p_1, p_2, p_3, p_4, p_5, p_6, p_7) = \left(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\right).$$

[5]It is widely observed and considered that well-organized deliberation has a mediating effect on their opinions (e.g., Condorcet 1793; Fishkin 1991, 2009; Goeree and Yarive 2011; List, Luskin, Fishkin and McLean 2013).

[6]For each $p \in X^I$ and each $S \subset I$, we call a voter whose peak is a (left) median among p_S a *median voter* among p_S.

Then

$$p' = \left(\frac{2}{7}, \frac{1}{7}, \frac{5}{7}, \frac{4}{7}, \frac{5}{7}, \frac{2}{7}, 1\right) \in W(p, \{1,2,3,6\}),$$

$$p'' = \left(1, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{5}{7}, \frac{2}{7}\right) \in W(p, \{1,6,7\}),$$

$$p''' = \left(\frac{3}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{4}{7}, \frac{6}{7}\right) \in C(p, \{1,2,3,6,7\}).$$

2.2 Axioms

Our main axiom requires that a rule does not increase the distance between a peak profile of a deliberation group and a collective outcome with respect to the group's *after-deliberation preferences.*

Axiom 1 (Deliberation monotonicity)**.** A rule f is *deliberation monotonic* if for each $p \in X^I$, each $S \subset I$, and each $p' \in C(p, S)$,

$$d(p'_S, f(p')) \leq d(p'_S, f(p)). \tag{2}$$

The left hand side of (2) is the distance between p'_S and $f(p')$ that is the outcome chosen after deliberation. The right hand side of (2) is the distance between p'_S and $f(p)$ that would have occurred without deliberation. If a rule is not deliberation monotonic, the group may regret the deliberation. Consequently, if we would like to avoid raising a deliberation group's dissatisfaction after voting, we should use a deliberation monotonic rule to aggregate their preferences. The next axiom is a variant of deliberation monotonicity. It requires that a rule does not increase the distance between a peak profile of all voters and an outcome with respect to their after-deliberation preferences.

Axiom 2 (Total deliberation monotonicity)**.** A rule f is *totally deliberation monotonic* if for each $p \in X^I$, each $S \subset I$, and each $p' \in C(p, S)$,

$$d(p', f(p')) \leq d(p', f(p)).$$

The third axiom is stronger than the above two axioms, which requires that a rule does not increase the distance between each voter's peak and an outcome with respect to one's preference after deliberation.

Axiom 3 (Individual deliberation monotonicity)**.** A rule f is *individually deliberation monotonic* if for each $p \in X^I$, each $S \subset I$, each $p' \in W(p, S)$, and each $i \in I$,

$$d(p'_i, f(p')) \leq d(p'_i, f(p)).$$

Remark 1. *If a rule is individually deliberation monotonic, then it is deliberation monotonic and totally deliberation monotonic.*

The following axiom ensures that for each $p \in X^I$, there exists no alternative $y \in X$ such that all voters weakly prefer y to $f(p)$ and some voters strictly prefer y to $f(p)$. In other words,

Axiom 4 (Efficiency)**.** A rule f is *efficient* if for each $p \in X^I$,

$$f(p) \in [\min_{j \in I} p_j, \max_{j \in I} p_j].$$

The following axiom requires that a rule does not discriminate voters by their names.

Axiom 5 (Anonymity)**.** A rule f is *anonymous* if for each $p \in X^I$ and each permutation $\pi : I \to I$,

$$f(p) = f(p(\pi)),$$

where $p(\pi) = (p_{\pi(1)}, p_{\pi(2)}, \ldots, p_{\pi(n)})$.

The next axiom requires that a rule does not discriminate alternatives by their values but respects only a pattern of a profile.[7][8]

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[7]This neutrality is different from the standard neutrality condition defined by May (1952). Since there are order relations on alternatives in a spatial model, we cannot arbitrarily re-label alternatives. To preserve order on alternatives, we restrict the way of changing their names.

[8]One may consider that we do not have to treat an edge point and an interior point equally because those points are not homogeneous. This is true. However, our results do not depend the heterogeneity. Indeed, we can easily check that our results hold even if $X = \mathbb{R}$ from our proof.

Axiom 6 (Neutrality)**.** A rule f is *neutral* if for each $p \in X^I$ and each $\alpha \in \mathbb{R}$,

$$f(p + \overline{\alpha}) = f(p) + \alpha,$$

as long as $p + \overline{\alpha} = (p_1 + \alpha, p_2 + \alpha, \dots, p_n + \alpha) \in X^I$ and $f(p) + \alpha \in X$.

3 Main Results in a Spatial Model

We present a particular form of distance functions: for each $p \in X^I$, each $f(p) \in X$, and each $S \subset I$,

$$d(p_S, f(p)) = \sum_{i \in S} |p_i - f(p)|.$$

We call this form of d the *Bordian distance function.*[9] In this section, we assume that d is Bordian. In Section 5, we discuss more general distance functions satisfying (1). The following proposition states that all k-th rules are monotonic for any centering transformation of preferences.

Proposition 1. *For each $k \in \{1, 2, \dots, n\}$, the k-th rule is deliberation monotonic.*

Proof. See Appendix A. □

By Proposition 1, a median rule is deliberation monotonic. In addition, a median rule also satisfies the following property, which is much stronger than total deliberation monotonicity.

Axiom 7 (Total deliberation monotonicity for all deliberations)**.** A rule f is *totally deliberation monotonic for all deliberations* if for each $p, p' \in X^I$ and each $S \subset I$,

$$d(p', f(p')) \leq d(p', f(p)).$$

Proposition 2. *A median rule is totally deliberation monotonic for all deliberations.*

[9]This form of d is related to the *inverse Borda count* in a spatial model, as defined by Feld and Grofman (1988). In a one dimensional space, $d(p_i, f(p)) = |p_i - f(p)|$ is very close to the inverse Borda count of $f(p)$ by i.

Proof. See Appendix A. □

Clearly, a median rule is totally deliberation monotonic. The median rule is known to satisfy desirable properties when preferences are single-peaked or Euclidean (Black 1948a,b; Arrow 1951; Moulin 1980). We show that the rule also satisfies deliberation monotonicity axioms. We have found again that the rule is one of the most compelling rules in a spatial model with Euclidean preferences.

Let us examine other rules that satisfy deliberation monotonicity. The *mean rule* is a simple voting rule, which is defined as for each $p \in X^I$,

$$f^{mean}(p) \equiv \frac{\sum_{i \in I} p_i}{|I|}.$$

Example 1 (The mean rule is not deliberation monotonic.)**.** Let $I = \{1,2,3\}$ and $p_1 = 0, p_2 = \frac{1}{3}, p_3 = 1$. Then $f^{mean}(p) = \frac{1}{3} \cdot \left(0 + \frac{1}{3} + 1\right) = \frac{4}{9}$. Let $S = \{1,2\}$. Let $p' \in C(p,S)$ be such that $(p'_1, p'_2) = \left(\frac{1}{3}, \frac{1}{3}\right)$ and $p'_3 = p_3$. Then $f^{mean}(p') = \frac{1}{3} \cdot \left(\frac{1}{3} + \frac{1}{3} + 1\right) = \frac{5}{9}$. Let us compare the distances from p'_S to $f^{mean}(p)$ and to $f^{mean}(p')$.

$$\begin{aligned} d(p'_S, f^{mean}(p')) &= \left|\frac{1}{3} - \frac{5}{9}\right| + \left|\frac{1}{3} - \frac{5}{9}\right| = \frac{4}{9}, \\ d(p'_S, f^{mean}(p)) &= \left|\frac{1}{3} - \frac{4}{9}\right| + \left|\frac{1}{3} - \frac{4}{9}\right| = \frac{2}{9}. \end{aligned}$$

Therefore, $d(p'_S, f^{mean}(p')) > d(p'_S, f^{mean}(p))$, that is, f^{mean} is not deliberation monotonic. ♢

Example 2 (The mean rule is not totally deliberation monotonic.)**.** Let $I = \{1,2,3\}$ and $p_1 = 0, p_2 = \frac{1}{3}, p_3 = 1$. Then $f^{mean}(p) = \frac{1}{3} \cdot (0 + \frac{1}{3} + 1) = \frac{4}{9}$. Let $S = \{1,2\}$. Let $p' \in C(p,S)$ be such that $(p'_1, p'_2) = (\frac{1}{3}, \frac{1}{3})$ and $p'_3 = p_3$. Then $f^{mean}(p') = \frac{1}{3} \cdot (\frac{1}{3} + \frac{1}{3} + 1) = \frac{5}{9}$. Let us compare the distances from p' to $f^{mean}(p)$

and to $f^{mean}(p')$.

$$d(p'_S, f^{mean}(p')) = \left|\frac{1}{3} - \frac{5}{9}\right| + \left|\frac{1}{3} - \frac{5}{9}\right| + \left|1 - \frac{5}{9}\right| = \frac{8}{9},$$
$$d(p'_S, f^{mean}(p)) = \left|\frac{1}{3} - \frac{4}{9}\right| + \left|\frac{1}{3} - \frac{4}{9}\right| + \left|1 - \frac{4}{9}\right| = \frac{7}{9}.$$

Therefore, $d(p', f^{mean}(p')) > d(p', f^{mean}(p))$, that is, f^{mean} is not totally deliberation monotonic. $\diamondsuit$

Next, we check if any other rule satisfies variants of deliberation monotonicity and other standard axioms. First, we show that the leftest rule and the rightest rule are the only rules that satisfy efficiency, anonymity, neutrality, and individual deliberation monotonicity.

Theorem 1. *A rule satisfies efficiency, anonymity, neutrality, and individual deliberation monotonicity if and only if it is either the leftest rule or the rightest rule.*

Proof. See Appendix B. $\square$

The leftest rule may look peculiar but it can be interpreted as the "unanimity rule" in situations where people make a collective decision on monetary valuation. For example, in an environmental monetary valuation (e.g.; Spash 2007), p_i represents i's willingness to pay to preserve environmental assets. The leftest peak is the smallest willingness to pay, so the leftest peak represents the largest amount of money that is unanimously agreed to be paid. Moreover, in a binary choice model, where $X = \{0, 1\}$ and 0 stands for "acquit" and 1 stands for "convict", the leftest rule is equivalent to the unanimity rule. In Section 4, we suggest that the counterpart of Theorem 1 holds in a binary choice model.

By Theorem 1, the leftest rule and the rightest rule are deliberation monotonic and totally deliberation monotonic. In addition, by Proposition 1 and 2, the median rules are deliberation monotonic and totally deliberation monotonic. We next determine if other k-th rules are totally deliberation monotonic. In fact, if a k-th rule satisfies total deliberation monotonicity, then it is either one of the leftest rule, the rightest rule, and the median rules.

Proposition 3. *When n is odd, if a k-th rule is totally deliberation monotonic, then either $k = 1$, n, or $\frac{n+1}{2}$. When n is even, if a k-th rule is totally deliberation monotonic, then either $k = 1$, n, $\frac{n}{2}$, or $\frac{n+2}{2}$.*

Proof. By Theorem 1, Proposition 1 and 2, f^L, f^R, f^{LM}, and f^{RM} are totally deliberation monotonic. We shall show that other k-th rules are not when $n \geq 5$.[10]

Case 1 (n is odd). Let $n \geq 5$ and take any $k \in \{2, \ldots, \frac{n-1}{2}, \frac{n+3}{2}, \ldots, n-1\}$. Let f be the k-th rule. We will show that f is not totally deliberation monotonic. Let $p \in X^I$ be such that $p_1 = \frac{1}{n}, p_2 = \frac{2}{n}, \ldots, p_{n-1} = \frac{n-1}{n}, p_n = 1$. Since f is the k-th rule, $f(p) = p_k$.

Consider the case with $2 \leq k \leq \frac{n-1}{2}$. The proof for the case with $\frac{n+3}{2} \leq k \leq n-1$ is parallel. Let $p' \in C(p, \{k-1, k\})$ be such that

$$\begin{aligned} p'_k &= p_k - \frac{1}{2n}, \\ p'_{k-1} &= p_{k-1}. \end{aligned}$$

Then

$$\begin{aligned} d(p', f(p')) - d(p', f(p)) &= k\left(-\frac{1}{2n}\right) + (n-k)\left(\frac{1}{2n}\right) \\ &= \frac{1}{2n}(n-2k) \\ &\geq \frac{1}{2n}\left(n - 2\frac{n-1}{2}\right) \qquad \text{since } k \leq \frac{n-1}{2} \\ &= \frac{1}{2n} > 0. \end{aligned}$$

Thereore f is not totally deliberation monotonic.

Case 2 (n is even). Let $n \geq 6$ and take any $k \in \{2, \ldots, \frac{n-2}{2}, \frac{n+4}{2}, \ldots, n-1\}$. Let f be the k-th rule. We will show that f is not totally deliberation monotonic. Let $p \in X^I$ be such that $p_1 = \frac{1}{n}, p_2 = \frac{2}{n}, \ldots, p_{n-1} = \frac{n-1}{n}, p_n = 1$. Since f is the k-th rule, $f(p) = p_k$.

Consider the case with $2 \leq k \leq \frac{n-2}{2}$. The proof for the case with $\frac{n+4}{2} \leq k \leq$

[10] If $n \leq 4$, there does not exist a k-th rule except for f^1, f^n, f^{LM}, and f^{RM}.

$n-1$ is parallel. Let $p' \in C(p, \{k-1, k\})$ be such that

$$p'_k = p_k - \frac{1}{2n},$$
$$p'_{k-1} = p_{k-1}.$$

Then

$$\begin{aligned}
d(p', f(p')) - d(p', f(p)) &= k\left(-\frac{1}{2n}\right) + (n-k)\left(\frac{1}{2n}\right) \\
&= \frac{1}{2n}(n-2k) \\
&\geq \frac{1}{2n}\left(n - 2\frac{n-2}{2}\right) \qquad \text{since } k \leq \frac{n-2}{2} \\
&= \frac{1}{n} > 0.
\end{aligned}$$

Thereore f is not totally deliberation monotonic. □

Next, we examine a property of k-th rules from another stand point. An essential axiom here is *strategy-proofness.* If a rule satisfies strategy-proofness, then for each preference profile and each voter, there exists no incentive to misreport his preference.

Axiom 8 (Strategy-proofness)**.** A rule f is *strategy-proof* if for each $p \in X^I$, each $i \in I$, and each $p'_i \in X$,

$$|f(p) - p_i| \leq |f(p'_i, p_{-i}) - p_i|.$$

Moulin (1980) shows that the *generalized median rules*, which contain all k-th rules, are the only rules that satisfy efficiency, anonymity, and strategy-proofness.

Theorem 2 (Moulin 1980)**.** *The following two statements are equivalent:*

(i) a rule f satisfies efficiency, anonymity, and strategy-proofness;

(ii) a rule f is a generalized median rule, that is, there exists $a = (a_1, a_2, \ldots, a_{n-1}) \in$

X^{n-1} *such that for each* $p \in X^I$,

$$f(p) = med(p, a),$$

where $med(p, a)$ *denotes the median among a peak profile* $(p, a) \in X^{2n-1}$.

In addition, we can prove that if a generalized median rule satisfies *neutrality*, then it is either one of the k-th rules. Namely, a rule satisfies efficiency, anonymity, strategy-proofness, and neutrality if and only if it is either one of the k-th rules. Moreover, combining the results of Proposition 3, we get the following characterization result.

Theorem 3. *A rule satisfies efficiency, anonymity, strategy-proofness, neutrality, and total deliberation monotonicity if and only if it is either one of the leftest rule, rightest rule, left median rule, and the right median rule.*

Proof. See Appendix B. □

Interestingly, a rule choosing a moderate point and a rule choosing an extreme point in each Pareto set are characterized by the same set of axioms, while any other rule is not. Moreover, in the above theorem, we can replace *total deliberation monotonicity* with a stronger condition. The total deliberation monotonicity is defined for the centering transformations but the next variant is based on weakly centering transformations.[11]

Axiom 9 (Strong total deliberation monotonicity)**.** A rule f is *strongly totally deliberation monotonic* if for each $p \in X^I$, each $S \subset I$, and each $p' \in W(p, S)$,

$$d(p', f(p')) \leq d(p', f(p)).$$

Corollary 1. *A rule satisfies efficiency, anonymity, strategy-proofness, neutrality, and strong total deliberation monotonicity if and only if it is either one of the leftest rule, rightest rule, left median rule, and the right median rule.*

[11]We can rank three variants of the total deliberation monotonicity axiom as follows; the strongest is the *total deliberation monotonicity for all deliberation* (Axiom 7), the next is the *strong total deliberation monotonicity* (Axiom 9), and the weakest is the *total deliberation monotonicity* (Axiom 2).

4 Binary Choice Model

As mentioned in Section 1, many previous studies on deliberative collective decision-making are conducted on jury models (e.g., Couglan 2000; Austen-Smith and Fedderson 2005, 2006; Gerardi and Yariv 2007). In this section, we determine if our results hold in a binary choice model.

Let $X = \{0, 1\}$, where 0 stands for *acquit* and 1 stands for *convict*. Each voter has a Euclidean preference on X. In this section, we use the Bordian form of d. Since X is binary, $d(p_i, x) = 0$ if and only if $p_i = x$, and $d(p_i, x) = 1$ if and only if $p_i \neq x$ for each $p_i \in X$ and $x \in X$. Therefore, the distance measured by the Bordian distance function is the inverse Borda count in the original sense.

On the domain, a weakly centering transformation is rewritten more simply. For preference profiles $p, p' \in X^I$ and a group $S \subset I$, call p' a *weakly centering transformation* of p at S if

$$[\forall i, j \in S, p_i = p_j] \Longrightarrow [\forall i \in S, p'_i = p_i],$$

and for all $i \notin S$, $p'_i = p_i$. This transformation only requires that if all voters' preferences are identical in a group, then *after-deliberation* preferences of the group do not change. A *centering* transformation is also rewritten as follows: for preference profiles $p, p' \in X^I$ and a group $S \subset I$, call p' a *centering transformation* of p at S if

$$p_i \neq p'_i \Longrightarrow p'_i = med(p'_S),$$

and for all $i \notin S$, $p'_i = p_i$, where $med(p'_S)$ denotes a median among p'_S. This transformation requires that if some voters' preferences change, then their ex-post peaks equal the ex-post majority.

In the binary choice model, the k-th rules can be interpreted as variants of the majority rule. For example, if n is odd and $k = \frac{n+1}{2}$, then f^k is the simple majority rule. If $k = 1$, then f^k is the unanimity rule because $f^1(p) = 1$ if and only if $p_i = 1$ for all $i \in I$. Moreover, if $k = \frac{1}{3}n \in \mathbb{N}$, then f^k is the $\frac{2}{3}$-majority rule. The $\frac{2}{3}$-majority rule is used to make a critical decision such as a revision of the constitution in Japan.[12] We show that these variants of the majority rules are deliberation monotonic in the binary choice model. In addition, we can obtain other counterparts of our results in Section 3.

Theorem 4. *On the binary choice domain,*

(i) all k-th rules are deliberation monotonic.

(ii) f^L, f^R, f^{LM}, and f^{RM} are totally deliberation monotonic but other k-th rules are not.

(iii) a rule satisfies efficiency, anonymity, and individual deliberation monotonicity if and only if it is either the leftest rule or the rightest rule.[13]

Proof. We only present a proof for (i). Other proofs can be done similarly to the proofs of counterparts in Section 3. Take any $k \in \{1, 2, \ldots, n\}$. Let f be the k-th rule. Take any $p \in X^I$ and $S \subset I$ with $|S| \geq 2$. Consider the case with $f(p) = 0$. The proof for the case with $f(p) = 1$ is symmetric.

Take any $p' \in C(p, S)$. If $f(p) = f(p')$, then $d(p'_S, f(p)) = d(p'_S, f(p'))$. Consider the case with $f(p) < f(p') = 1$. Since $f(p) < f(p') = 1$, there exists $j \in S$ such that $p_j < p'_j = 1$. Since p' is a centering transformation of p at S, $med(p'_S) = 1$. Let $T \equiv \{k \in S : p'_k = 1\}$. Then $|T| \geq |S|/2$. Therefore,

$$d(p'_S, f(p')) = |T| \cdot d(1, 1) + (|S| - |T|) \cdot d(0, 1) \leq \frac{|S|}{2}.$$

On the other hand,

$$d(p'_S, f(p)) = |T| \cdot d(1, 0) + (|S| - |T|) \cdot d(0, 0) \geq \frac{|S|}{2}.$$

Therefore, f is deliberation monotonic. □

[12]If $k < \frac{n}{2}$, we call f^k a *super majority rule*. Super majority rules are often used in constitutional amendments; for example, the $\frac{2}{3}$-majority rule (the $\frac{1}{3}n$-th rule) is also used in the United States and the $\frac{3}{5}$-majority rule (the $\frac{2}{5}n$-th rule) is used in Fance. Caplin and Nalebuff (1988) study properties of super majority rules, in particular, the 64% majority rule in a spatial model with Euclidean preferences.

[13]Coughlan (2000) shows that the unanimity rule performs better than other alternative rules when communication is permitted. Our result gives another superiority of the unanimity rule.

The next theorem is not a counterpart of results when $X = [0, 1]$. We can more simply characterize k-th rules in the binary choice model.[14]

Theorem 5. *A rule is efficient, anonymous, and deliberation monotonic if and only if it is one of the k-th rules.*

Proof. It is clear that all k-th rules are efficient and anonymous. By Theorem 4-(i), they are also deliberation monotonic.

We shall show that k-th rules are the only rules satisfying these axioms. Let f be a rule that satisfies efficiency, anonymity, and deliberation monotonicity. Define

$$\begin{aligned}
p^0 &\equiv (0, 0, \ldots, 0) \in X^I, \\
p^1 &\equiv (1, 0, \ldots, 0) \in X^I, \\
p^2 &\equiv (1, 1, 0, \ldots, 0) \in X^I, \\
&\vdots \\
p^k &\equiv (\underbrace{1, 1, \ldots, 1}_{k}, 0, \ldots, 0) \in X^I, \\
&\vdots \\
p^n &\equiv (1, 1, \ldots, 1) \in X^I.
\end{aligned}$$

By efficiency, $f(p^0) = 0$ and $f(p^n) = 1$.

- If $f(p^k) = 0$ for all $k \in \{1, 2, \ldots, n-1\}$, then, by anonymity, f is f^L.
- If $f(p^k) = 1$ for all $k \in \{1, 2, \ldots, n-1\}$, then, by anonymity, f is f^R.

Consider the other case:

$$\begin{cases} \exists k \in \{1, 2, \ldots, n-1\},\ f(p^k) = 1, \text{ and} \\ \exists \ell \in \{1, 2, \ldots, n-1\},\ f(p^\ell) = 0. \end{cases}$$

[14] This result is related to May (1952). He characterizes the simple majority ordering rule by efficiency, anonymity, neutrality, and *positive respoinsiveness* in a binary choice situation.

Note that $k \neq \ell$.

We shall show $\ell < k$. Suppose not, $k < \ell$. Then there exists $j \in \{1, \ldots, n-1\}$ with $k \leq j \leq \ell$ such that $f(p^j) = 1$ and $f(p^{j+1}) = 0$. Let $p \equiv p^j$.

Since $1 \leq j$ and $j + 1 < n$,

$$1 \leq |\{i \in I : p_i = 1\}| \leq n - 2 \text{ and } \ 2 \leq |\{i \in I : p_i = 0\}|.$$

Take any $a, b \in \{i \in I : p_i = 0\}$ and $c \in \{i \in I : p_i = 1\}$ and let $S = \{a, b, c\}$. Note that $p_S = (p_a, p_b, p_c) = (0, 0, 1)$. Let $p' \in C(p, S)$ be such that $p'_S = (0, 1, 1)$. By anonymity, $f(p') = f(p^{j+1}) = 0$. However, since $f(p) = 0$,

$$d(p'_S, f(p')) = d(0, 0) + d(1, 0) + d(1, 0) = 2,$$

$$d(p'_S, f(p)) = d(0, 1) + d(1, 1) + d(1, 1) = 1,$$

a contradiction to deliberation monotonicity.

Therefore, $\ell < k$ holds. Moreover, from the above argument, there exists a unique $k' \in \{\ell + 1, \ell + 2, \ldots, k\}$ such that $f(p^{k'}) = 1$, $f(p^{k'-1}) = 0$ and $f(p^i) = 1$ for each $i \geq k'$. Let $\alpha \equiv n - k' - 1$. By anonymity, f is the α-th rule.

Since k' is equal to either 2 or 3 or $\cdots$ or $(n-1)$, α is either 2 or $\cdots$ or $(n-1)$. Because f^L is the 1st rule and f^R is the n-th rule, f is either one of the k-th rules. □

5 General Distance Functions

In this section, we consider the general distance functions that satisfy all three properties in (1) on $X = [0, 1]$. Desirable properties of median rules depend on the Bordian distance function. However, the leftest rule and the rightest rule still satisfy individual deliberation monotonicity for every weakly centering deliberation, even with a general form of d. Conversely, for each distance d, both rules are the only rules satisfying efficiency, anonymity, neutrality, and individual deliberation monotonicity for every weakly centering deliberation. Consequently, the result of Theorem 1 is true for every distance function.[15]

Corollary 2. *Let d be any distance function satisfying (1). A rule satisfies efficiency, anonymity, neutrality, and individual deliberation monotonicity if and only if it is either the leftest rule or the rightest rule.*

Proof. A proof is same as that of Theorem 1. □

6 Conclusion

We searched for voting rules that can appropriately reflect the changes in voters' preferences in a one-dimensional spatial model. As a main result, we showed that the median rule and the unanimity-type rules are the only rules that satisfy the deliberation monotonicity axioms and other standard axioms. As in past studies, we also found that the median rule is one of the most compelling rules when preferences are Euclidean. However, we showed that the unanimity-type rules, that is, the leftest rule and the rightest rule, are the only rules that satisfy desirable properties in the general setting.

From the view point of social choice theory, deliberative democracy essentially consists of two elements: preference transformation by deliberation and preference aggregation by voting. Our analysis formally deals these two elements and offers insights on designing voting rules for deliberative democracy. In this paper, we focused on a one-dimensional issue space, and extending our analysis to cases with a multi-dimensional issue space remains future research.

Appendix A

Proof of Proposition 1. Take any $p \in X^I$, any $S \subset I$ and any $k \in \{1, 2, \ldots, n\}$ and let f be the k-th rule. Take any $p' \in C(p, S)$. The proof proceeds in two steps.

[15] From the proof of the theorem, we can find that the result holds with *single-peaked preferences*, but we do not discuss this scenario.

Step 1. We shall show that

$$f(p) < f(p') \Longrightarrow \begin{cases} |\{i \in S : p'_i < f(p')\}| \leq \frac{|S|-1}{2} & \text{if } |S| \text{ is odd,} \\ |\{i \in S : p'_i < f(p')\}| \leq \frac{|S|}{2} & \text{if } |S| \text{ is even.} \end{cases}$$

Suppose $f(p) < f(p')$. Let us consider the case where $|S|$ is odd. The proof for the case where $|S|$ is even can be proven similarly. Let $M \equiv \{i \in I : p_i = f(p)\}$. Consider the case with $M \cap S = \emptyset$. Since $f(p)$ is the k-th peak among $p_1, p_2, \ldots, p_n$ and $f(p) < f(p')$, there exists $j \in S$ such that $p_j < f(p)$ and $p'_j > f(p)$. Moreover, since such j exists and p' is a centering transformation of p at S,

$$\max_{k \in S}\{p_k : p_k < f(p)\} \leq p_S\left[\frac{|S|+1}{2}\right].$$

Therefore,

$$1 \leq |\{i \in S : p_i < f(p)\}| \leq \frac{|S|+1}{2}.$$

Letting $k = |\{i \in S : p_i < f(p) \text{ and } p'_i \geq f(p)\}|$,

$$|\{i \in S : p'_i < f(p)\}| \leq \frac{|S|+1}{2} - k. \tag{3}$$

Therefore,

$$|\{i \in S : f(p) \leq p'_i < f(p')\}| \leq k - 1. \tag{4}$$

By (3) and (4), since $\{i \in S : p'_i < f(p)\} \cap \{i \in S : f(p) \leq p'_i < f(p')\} = \emptyset$,

$$|\{i \in S : p'_i < f(p')\}| \leq \frac{|S|+1}{2} - k + k - 1 = \frac{|S|-1}{2}.$$

The case with $M \cap S \neq \emptyset$ can be proven similarly.

Step 2. We shall show $d(p'_S, f(p')) \leq d(p'_S, f(p))$. Let us consider the case where $|S|$ is odd. The proof for the case where $|S|$ is even can be proven similarly. If $f(p') = f(p)$, then $d(p'_S, f(p')) = d(p'_S, f(p))$. Consider the case with $f(p) < f(p')$.

The proof for the case with $f(p) > f(p')$ is parallel, so we omit it. By Step 1,

$$|\{i \in S : p'_i < f(p')\}| \leq \frac{|S|-1}{2},$$

so that

$$|\{i \in S : p'_i \geq f(p')\}| \geq \frac{|S|+1}{2}.$$

Let

$$\begin{aligned}
A &\equiv \{i \in S : p'_i < f(p)\}, \\
B &\equiv \{i \in S : f(p) \leq p'_i < f(p')\}, \\
C &\equiv \{i \in S : f(p') \leq p'_i\}.
\end{aligned}$$

Note that A, B and C are disjoint and $A \cup B \cup C = S$. Also,

$$\begin{aligned}
|A \cup B| = |A| + |B| &\leq \frac{|S|-1}{2}, \\
|C| &\geq \frac{|S|+1}{2}.
\end{aligned}$$

Let $\delta \equiv d(f(p), f(p')) > 0$. Since d is Bordian,

$$d(p'_S, f(p')) = d(p'_A, f(p')) + d(p'_B, f(p')) + d(p'_C, f(p')). \tag{5}$$

The terms in the left hand side of (5) are

$$\begin{aligned}
d(p'_A, f(p')) &= \sum_{i \in A} d(p'_i, f(p')) \\
&= \sum_{i \in A} \left(d(p'_i, f(p)) + d(f(p), f(p'))\right) \\
&= d(p'_A, f(p)) + |A|\delta,
\end{aligned}$$

$$
\begin{aligned}
d(p'_B, f(p')) &= \sum_{i\in B} d(p'_i, f(p')) \\
&= \sum_{i\in B} \left(d(f(p), f(p')) - d(p'_i, f(p))\right) \\
&= |B|\delta - d(p'_B, f(p)) \\
&\leq |B|\delta + d(p'_B, f(p)),
\end{aligned}
$$

$$
\begin{aligned}
d(p'_C, f(p')) &= \sum_{i\in C} d(p'_i, f(p')) \\
&= \sum_{i\in C} \left(d(p'_i, f(p)) - d(f(p), f(p'))\right) \\
&= d(p'_C, f(p)) - |C|\delta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d(p'_S, f(p')) &= d(p'_A, f(p')) + d(p'_B, f(p')) + d(p'_C, f(p')) \\
&\leq d(p'_A, f(p)) + |A|\delta + |B|\delta + d(p'_B, f(p)) + d(p'_C, f(p)) - |C|\delta \\
&= d(p'_S, f(p)) + (|A| + |B| - |C|)\,\delta \\
&< d(p'_S, f(p)).
\end{aligned}
$$

Therefore, the k-th rule is deliberation monotonic. $\square$

Proof of Proposition 2. Take any $p \in X^I$. Without loss of generality, we assume

$$p_1 \leq p_2 \leq \cdots \leq p_n.$$

Case 1 (n is odd). It suffices to show that

$$\{f^M(p)\} = \arg\min_{x\in X} \sum_{i\in I} |x - p_i|. \tag{6}$$

For each $k = 1, 2, \ldots, \dfrac{n-1}{2}$ and each $x \in X$, let

$$\delta_k(x) \equiv |x - p_k| + |x - p_{n-k+1}|.$$

Then

$$\delta_1(x) = |x - p_1| + |x - p_n| = \begin{cases} 2x - p_1 - p_n & \text{if } x \geq p_n, \\ p_1 - p_n & \text{if } p_1 \leq x \leq p_n, \\ p_n + p_1 - 2x & \text{if } x \leq p_1. \end{cases}$$

If $x \geq p_n$, then $x - p_n + x - p_1 \geq p_n - p_n + p_n - p_1 = p_n - p_1 \geq p_1 - p_n$. If $x \leq p_1$, then $p_n - x + p_1 - x \geq p_n - p_1 + p_1 - p_1 = p_n - p_1 \geq p_1 - p_n$. Therefore, $\delta_1(x)$ is minimized at each $x \in [p_1, p_n]$. Similarly, $\delta_k(x)$ is minimized at each $x \in [p_k, p_{n-k+1}]$ for each $k = 1, 2, \ldots, \dfrac{n-1}{2}$. Let $\delta_{\frac{n+1}{2}}(x) \equiv |x - p_{\frac{n+1}{2}}|$. It is minimized at $x = p_{\frac{n+1}{2}}$. Note that $[p_1, p_n] \supset [p_2, p_{n-1}] \supset \cdots \supset [p_{\frac{n-1}{2}}, p_{\frac{n+3}{2}}] \supset \{p_{\frac{n+1}{2}}\} = \{f^M(p)\}$. Since $\sum_{i \in I} |x - p_i| = \delta_1(x) + \delta_2(x) + \cdots + \delta_{\frac{n+1}{2}}$, it is minimized at $x = f^M(p)$. We established (6).

Case 2 ($\boldsymbol{n}$ is even). It suffices to show that

$$[f^{LM}(p), f^{RM}(p)] = \arg\min_{x \in X} \sum_{i \in I} |x - p_i|. \tag{7}$$

For each $k = 1, 2, \ldots, \dfrac{n}{2}$ and each $x \in X$, let

$$\delta_k(x) \equiv |x - p_k| + |x - p_{n-k+1}|.$$

Similarly to the case where n is odd, $\delta_k(x)$ is minimized at each $x \in [p_k, p_{n-k+1}]$ for each $k = 1, 2, \ldots, \dfrac{n}{2}$. Since $[p_{\frac{n}{2}}, p_{\frac{n}{2}+1}] = [f^{LM}(p), f^{RM}(p)]$, we establised (7).

By (6) and (7), a median rule is totally deliberation monotonic. □

Appendix B

Proof of Theorem 1

Note that $X = [0, 1]$. We first show that the leftest rule and the rightest rule are individually deliberation monotonic for each distance function.

Lemma 1. *For each distance d, the leftest rule and the rightest rule are individually deliberation monotonic.*

Proof. Let d be any distance function satisfying (1). We only give a proof for the leftest rule because the proof for the rightest rule is parallel. Take any $p \in X^I$ and each $S \subset I$. Let $k \in I$ be a voter whose peak is equal to $f^L(p)$. Note that $p_k = \min_{i \in I} p_i$. Take any $p' \in W(p, S)$. Then $\{i \in I : p'_i < p_k\} = \emptyset$. Therefore, $f^L(p) = p_k \leq f^L(p')$. Since f^L is the leftest rule, $f^L(p') \leq p'_i$ for all $i \in I$. Hence, $f^L(p) \leq f^L(p') \leq p'_i$ for all $i \in I$,. Therefore, for any d, $d(p'_i, f^L(p')) \leq d(p'_i, f^L(p))$ for all $i \in I$. □

By Lemma 1, the leftest and the rightest rule satisfy individual deliberation monotonicity. It is clear that these rules are also efficient, anonymous, and neutral.

Conversely, let us consider any rule f that satisfies efficiency, anonymity, neutrality, and individual deliberation monotonicity. We shall show that either $f(p) = p[1]$ for all $p \in X^I$ or $f(p) = p[n]$ for all $p \in X^I$. We denote the set of peaks by $T(p) \equiv \{p_1, p_2, \ldots, p_n\}$ for each $p \in X^I$. Our proof consists of four claims.

Claim 1. For each $p \in X^I$, each $S \subset I$, and each $p' \in W(p, S)$, if there exists $k \in I$ with $p'_k = f(p)$, then $f(p) = f(p')$.

Proof. Take any $p \in X^I$, any $S \subset I$, and any $p' \in W(p, S)$. Assume that there exists $k \in I$ with $p'_k = f(p)$. Then $d(p'_k, f(p)) = 0$. If $f(p') \neq f(p)$, then $d(p'_k, f(p')) > 0 = d(p'_k, f(p))$, a contradiction to individual deliberation monotonicity. Therefore, $f(p') = f(p)$. □

Claim 2. There exists no $p \in X^I$ with $f(p) \in T(p) \setminus \{p[1], p[n]\}$.

Proof. Suppose, by contradiction, that there exists $p \in X^I$ such that for some $k \in I$, $f(p) = p_k$ and $p[1] < p_k < p[n]$. Let $\ell \in I$ be a voter whose peak is equal to $p[n]$. Let $p' \in W(p, I)$ be such that

$$
\begin{aligned}
&p'_i = p[1] \quad \text{if } i \in I \setminus \{k, \ell\},\\
&p'_\ell = \begin{cases} p_k + \frac{p_k - p[1]}{2} & \text{if } p_\ell - p_k \geq \frac{p_k - p[1]}{2},\\ p_\ell \ \text{ if } p_\ell - p_k < \frac{p_k - p[1]}{2}, \end{cases}\\
&p'_k = p_k.
\end{aligned}
$$

Let $\alpha \equiv p'_\ell - p'_k$. By Claim 1, $f(p) = f(p') = p'_k$. Let $q \in W(p', I)$ be such that

$$
\begin{aligned}
&q_i = p'[1] \quad \text{if } i \in I \setminus \{k, \ell\},\\
&q_\ell = p'_k - \alpha,\\
&q_k = p_k.
\end{aligned}
$$

Note that $q[1] < q_\ell < q_k$. By Claim 1, $f(q) = f(p') = q_k$. Let $r \equiv q + \alpha$. Then $r_\ell = p'_k$ and $r_k = p'_\ell$. By neutrality, $f(r) = f(q) + \alpha = r_k = p'_\ell$. Note that $p'[1] = q[1] \leq r[1]$. Let $p'' \in W(p', I)$ be such that

$$
\begin{aligned}
&p''_i = r[1] \quad \text{if } i \in I \setminus \{k, \ell\},\\
&p''_\ell = p'_\ell,\\
&p''_k = p'_k.
\end{aligned}
$$

By Claim 1, $f(p'') = p''_k = p'_k$. Note that

$$p''_k = p'_k = f(p'') = r_\ell \text{ and } p''_\ell = p'_\ell = f(r) = r_k. \tag{8}$$

Since $p'_k \neq p'_\ell$,

$$f(p'') \neq f(r). \tag{9}$$

On the other hand, by the construction of r and p'',

$$r_i = p''_i \quad \forall i \in I \setminus \{k, \ell\}. \tag{10}$$

By (8) and (10) and anonymity, $f(p'') = f(r)$, a contradiction to (9). Therefore, there is no $p \in X^I$ with $f(p) \in T(p) \setminus \{p[1], p[n]\}$. $\square$

Claim 3. There exists no $p \in X^I$ with $f(p) \in I \setminus T(p)$.

Proof. Suppose, by contradiction, that there exists $p \in X^I$ with $f(p) \in I \setminus T(p)$. Let $k \in I$ be a voter whose peak is equal to $p[1]$ and let $\ell \in I$ be a voter whose peak is equal to $p[n]$. By efficiency, $f(p) \in [p[1], p[n]]$. Therefore, since $n \geq 3$,

$$\text{there exists } i, j \in I \text{ such that } p_i < f(p) < p_j \text{ with } i \neq k \text{ or } j \neq \ell. \tag{11}$$

We shall show that (11) contradicts Claim 2 or individual deliberation monotonicity. Let $p' \in W(p, \{i, j\})$ be such that

$$(p'_i, p'_j) = \begin{cases} (f(p), p_j) & \text{if } i \neq k \\ (p_i, f(p)) & \text{if } i = k \text{ and } j \neq \ell \end{cases}$$

If $f(p') = f(p)$, then $f(p') \in S(p') \setminus \{p'[1], p'[n]\}$, which contradicts Claim 2. If $f(p) < f(p')$, then $d(p'_k, f(p')) > d(p'_k, f(p))$, which contradicts individual deliberation monotonicity. If $f(p') < f(p)$, then $d(p'_\ell, f(p')) > d(p'_\ell, f(p))$, which contradicts individual deliberation monotonicity. Therefore, there exists no $p \in X^I$ with $f(p) \in I \setminus T(p)$. $\square$

Claim 4. Either $f(p) = p[1]$ for each $p \in X^I$ or $f(p) = p[n]$ for each $p \in X^I$.

Proof. By efficiency and Claims 2 and 3,

$$\text{either } f(p) = p[1] \text{ or } p[n] \quad \forall p \in X^I.$$

Let us show that if there exists $p \in X^I$ with $f(p) = p[1]$, then for each $q \in X^I$, $f(q) = q[1]$, that is, f is the leftest rule. The proof for the rightest rule is parallel.

Assume that there exists $p \in X^I$ with $f(p) = p[1]$. Suppose, by contradiction, that there exists $q \in X^I$ such that $q \neq p$ and $f(q) = q[n]$. Let $i \in I$ be a voter whose peak is equal to $p[1]$ and let $j \in I$ be one whose peak is equal to $p[n]$. Similarly, let $k \in I$ be a voter whose peak is equal to $q[1]$ and let $\ell \in I$ be one whose peak is equal to $q[n]$.

Case 1 $(\boldsymbol{p[n]-p[1] \leq q[n]-q[1]})$.

Subcase 1 $(\boldsymbol{p[1] \leq q[1]})$.

Let $\delta \equiv q[n]-p[1] \geq 0$. Let $r \equiv p+\delta$. Then $r[n]=q[n]$ and $r[1] \geq q[1]$. Note that $f(r)=f(p)+\delta=p[1]+\delta=r[1]$ by additive neutrality. Let $q' \in W(q,I)$ be such that

$$\begin{aligned} q'_t &= r[1] \quad \text{if } t \in I \setminus \{\ell\}, \\ q'_\ell &= q_\ell. \end{aligned}$$

Let $r' \in W(r,I)$ be such that

$$\begin{aligned} r'_t &= r[1] \quad \text{if } t \in I \setminus \{j\}, \\ r'_j &= r_j. \end{aligned}$$

Then $q'_\ell = r'_j$ and $q'_t = r[1]$ for each $t \in I \setminus \{\ell\}$ and $r'_t = r[1]$ for each $t \in I \setminus \{\ell\}$. Therefore, by anonymity, $f(q')=f(r')$. On the other hand, By Claim 2, $f(q') = q'[n]$ and $f(r')=r'[1]$, that is, $f(q') \neq f(r')$, a contradiction.

Subcase 2 $(\boldsymbol{p[1] > q[1]}$ and $\boldsymbol{p[n] \geq q[n]})$.

Let $p' \in W(p,I)$ be such that

$$\begin{aligned} p'_t &= q[n] \quad \text{if } t \in I \setminus \{i\}, \\ p'_i &= p_i. \end{aligned}$$

Let $q' \in W(q,I)$ be such that

$$\begin{aligned} q'_t &= q[n] \quad \text{if } t \in I \setminus \{k,\ell\}, \\ q'_k &= p[1], \\ q'_\ell &= q_\ell. \end{aligned}$$

By anonymity, $f(p')=f(q')$. However, by Claim 1, $f(p')=p'_i=p'[1] \neq q'[n] = q'_\ell = f(q')$, a contradiction.

Subcase 3 $(\boldsymbol{p[1] > q[1]}$ and $\boldsymbol{p[n] < q[n]})$.

Let $\delta \equiv q[n] - p[n]$ and let $r \equiv p + \delta$. Then $r[n] = q[n]$ and $r[1] > q[1]$. Let $r' \in W(r, I)$ be such that

$$\begin{aligned} r'_t &= r[1] \quad \text{if } t \in I \setminus \{j\}, \\ r'_j &= r_j. \end{aligned}$$

Let $q' \in W(q, I)$ be such that

$$\begin{aligned} q'_t &= r[1] \quad \text{if } t \in I \setminus \{\ell\}, \\ q'_\ell &= q_\ell. \end{aligned}$$

Similarly, by anonymity, $f(r') = f(q')$ but Claim 1 implies $f(r') \neq f(q')$, a contradiction.

Case 2 $(\boldsymbol{p[n] - p[1] > q[n] - q[1]})$**.**

The proof strategy for Case 2 is parallel to that for Case 1, so we omit it. □

Hence, we have shown that if a rule f satisfies efficiency, anonymity, neutrality, and individual deliberation monotonicity, then it is either the leftest rule or the rightest rule. Since we only used properties of general distance functions listed in (1), this result holds for every distance function.

Proof of Theorem 3

Note that $X = [0, 1]$.

Claim 5. For each $k \in \{1, 2, \ldots, n\}$, the k-th rule is one of the generalized median rules.

Proof. Take any $k \in \{1, 2, \ldots, n\}$. Let $a \in X^{n-1}$ be such that

$$\begin{aligned} a_1 = a_2 = \cdots = a_{n-k} &= 0, \\ a_{n-k+1} = a_{n-k+2} = \cdots = a_{n-1} &= 1. \end{aligned}$$

Take any $p \in X^I$ and let $q \equiv (p, a) \in X^{2n-1}$. Then

$$f^k(p) = p_k = q[n - k + k] = q[n] = med(q) = med(p, a).$$

Therefore, all k-th rules are generalized median rules. $\square$

For each $a \in X^{n-1}$, we denote g^a the generalized median rule characterized by the vector a.

Claim 6. If g^a is neutral, then $a \in \{0, 1\}^{n-1}$.

Proof. We will prove the contraposition. Suppose that there exists $j \in \{1, 2, \ldots, n-1\}$ with $a_j \notin \{0, 1\}$. Let $n_0 \equiv |\{i \in \{1, 2, \ldots, n-1\} : a_i = 0\}|$. Let

$$\ell \in \underset{i \in \{1,2,\ldots,n-1\}}{\arg\min} \{a_i : a_i \notin \{0, 1\}\}.$$

Let $p \in X^I$ be such that

$$p_1 = p_2 = \cdots = p_{n-n_0-1} = \frac{a_\ell}{2},$$
$$p_{n-n_0} = p_{n-n_0+1} = \cdots = p_n = \frac{1 + a_\ell}{2}.$$

Since g^a is a generalized median rule, $g^a(p) = med(p, a) = a_\ell$. Let $b \equiv \min(\frac{a_\ell}{4}, \frac{1-a_\ell}{4})$ and $\overline{b} \equiv (b, b, \ldots, b) \in X^I$. Note that $p - \overline{b} \in X^I$. By $med(p - \overline{b}, a) = a_\ell$, $g^a(p - \overline{b}) = g^a(p)$. Therefore, g^a is not neutral. Hence, if g^a is neutral, then $a \in \{0, 1\}^{n-1}$. $\square$

Claim 7. If $a \in \{0, 1\}^{n-1}$, then g^a is one of the k-th rules.

Proof. Take any $a \in \{0, 1\}^{n-1}$ and any $p \in X^I$. Let $q \equiv (p, a) \in X^{2n-1}$. Let $n_0 \equiv |\{i \in \{1, 2, \ldots, n-1\} : a_i = 0\}|$. Then $g^a(p) = med(q) = q[n] = p[n - n_0]$, that is, g^a is the $(n - n_0)$-th rule. Since $1 \leq n - n_0 \leq n$, the rule is one of the k-th rules. $\square$

Clearly, all k-th rules are neutral. Therefore, by Claims 5, 6 and 7, a generalized median rule is neutral if and only if it is one of the k-th rules. Moreover, by Proposition 3 and Theorem 2, a rule satisfies efficiency, anonymity, strategy-proofness, neutrality, and total deliberation monotonicity if and only if it is either one of the leftest rule, rightest rule, left median rule, and the right median rule.

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