A decomposition of strategy-proofness

Nozomu Muto

Department of Economics,
Yokohama National University

79-3 Tokiwadai, Hodogaya-ku,
Yokohama 240-8501, Japan

nozomu.muto@gmail.com

Shin Sato

Faculty of Economics,
Fukuoka University

8-19-1 Nanakuma, Jonan-ku,
Fukuoka 814-0180, Japan

shinsato@adm.fukuoka-u.ac.jp

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Abstract

Strategy-proofness has been one of the central axioms in the theory of social choice. We find that *strategy-proofness* is decomposed into three axioms: *top-restricted AM-proofness*, *weak monotonicity*, and *individual bounded response*. Among them, *individual bounded respose* is not as defensible as the other two, and we present possibility results by excluding *individual bounded response* from *strategy-proofness*. One of the results supports the plurality rule which is the most widely used rule in practice.

Keywords: individual bounded response, nonmanipulability, strategy-proofness, top-restricted AM-proofness, weak monotonicity

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1 Introduction

Consider a situation in which a society is to choose an alternative based on the agents' preferences over alternatives according to some rule (social choice rule), and all logically possible preferences are agents' admissible preferences. One of the desirable properties of a rule is *strategy-proofness* which says that each agent does not have an incentive to misreport his preference.[1] Our motivating question is "What is *strategy-proofness*?" Of course, we know the definition of *strategy-proofness*. What we want to do is understanding *strategy-proofness* by decomposing it into several weaker axioms. The decomposition is interesting by itself. Moreover, as we will discuss shortly, this will help us find an unattractive part of *strategy-proofness*, and opens a door to formulate a new nonmanipulability condition.

A main result of this paper is a characterization of *strategy-proofness*. Specifically, the collection of three axioms, *top-restricted AM-proofness*, *weak monotonicity*, and *individual bounded response*, is shown to be equivalent to *strategy-proofness*, i.e., *strategy-proofness* is decomposed into these three axioms. *Top-restricted AM-proofness* says that each agent cannot change the social choice from the second preferred one to the most preferred one by reporting a false preference which is "adjacent" to the sincere one. Two preferences are adjacent if the difference between them is the ranks of one pair of consecutively ranked alternatives. *Weak monotonicity* says that lifting the position of the social choice in agents' preferences without any other change does not change the social choice. *Individual bounded response* says that the difference between the ranks of the social choices at adjacent preferences R_i and R'_i is at most one, provided that the other agents' preferences are the same. In the following, we discuss our result and contribution in detail.

It is well known that it is difficult to satisfy *strategy-proofness*. The most important result is the Gibbard–Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975); *strategy-proofness* and some other mild condition together imply dictatorship. Because dictatorship is not acceptable, *strategy-proofness* does not serve as

[1] See Barberà (2010) for a survey.

a useful criterion of nonmanipulability to select a nonmanipulable rule among the acceptable ones. Because *monotonicity* is equivalent to *strategy-proofness* (Muller and Satterthwaite, 1977), *monotonicity* is not a useful criterion, either.[2] On the other hand, most common rules satisfy *weak monotonicity*. Thus, in the passage from *strategy-proofness* (equivalently, *monotonicity*) to *weak monotonicity*, impossibility changes to possibility. Then, what is the difference between *strategy-proofness* and *weak monotonicity*?

According to our result, the difference consists of the two axioms, *top-restricted AM-proofness* and *individual bounded response*. Let *weak monotonicity* be the starting point of looking for desirable rules. Then, there are many rules satisfying it. However, by additionally imposing *top-restricted AM-proofness* and *individual bounded response*, we reach *strategy-proofness*, and impossibility appears. Thus, if we take it for granted that each acceptable rule satisfies *weak monotonicity*, the cause of impossibility is *top-restricted AM-proofness* and *individual bounded response*.

We believe that *individual bounded response* is not as defensible as the other two. By removing *individual bounded response* from *strategy-proofness*, we have a weaker nonmanipulability condition which might be a useful criterion of nonmanipulability. In Section 3.4, we consider the class of scoring rules. By definition, they satisfy *weak monotonicity*. We find the class of scoring rules satisfying *top-restricted AM-proofness* and *top-restricted strategy-proofness*, respectively.[3] Each scoring rule satisfies *top-restricted AM-proofness* if and only if the score assigned to the second rank and the score assigned to the third rank are the same (Proposition

[2] *Monotonicity* says that expanding the lower contour set of the social choice in agents' preferences does not change the social choice. This property is often called *Maskin monotonicity*.

[3] *Top-restricted strategy-proofness* excludes the possibility of changing the social choice from the second ranked to the top ranked one, and the options for misrepresentation are not restricted. We can use *top-restricted strategy-proofness* instead of *top-restricted AM-proofness* in our decomposition theorem. (See Section 3.2.)

3.1). Thus, we always have a scoring rule satisfying *top-restricted AM-proofness*, and when there are many alternatives, we have many types of such scoring rules. Considering that *strategy-proofness* is violated by all scoring rules, *top-restricted AM-proofness* is a weak axiom. Proposition 3.2 shows that *top-restricted strategy-proofness* characterizes the plurality rule. The plurality rule, which is the most widely used rule in our lives, is theoretically supported by our analysis.

In sum, our contributions are the following; By decomposing *strategy-proofness*, we find its components. Based on that, we consider weaker nonmanipulability conditions than *strategy-proofness*. Then, we have practically interesting results. Especially, the plurality rule is supported by one of our results.

The rest of the paper is organized as follows. Section 2 introduces notation, and defines axioms on a rule. Section 3.1 presents our main result and its proof, Section 3.2 discusses the main result, Section 3.3 shows that the three axioms, *top-restricted AM-proofness*, *weak monotonicity*, and *individual bounded response*, are independent from each other, and Section 3.4 shows how our main theorem can be applied. Section 4 gives concluding remarks.

2 Basic notation and definitions

Let X be a finite set of *alternatives* with $|X| = m \geq 3$, $N = \{1, \dots, n\}$ be a finite set of *agents* with $n \geq 2$, $\mathcal{L}$ be the set of all *linear orders* on X. For each $R_i \in N$ and $x \in X$, let $\rho_{R_i}(x)$ be the rank of x at R_i, and $r^k(R_i)$ be the kth ranked alternative according to R_i. A function f from $\mathcal{L}^N$ into X is called a *rule*. We say that $x \in X$ *raises its position in the passage from R_i to R_i'*$(\neq R_i)$ if for each $y, z \in X \setminus \{x\}$, $y \; R_i \; z$ if and only if $y \; R_i' \; z$ and for each $w \in X$, $x \; R_i \; w$ implies $x \; R_i' \; w$. Alternatives x and y are *adjacent in $R_i \in \mathcal{L}$* if they are consecutively ranked in R_i, i.e., there is no $z \in X \setminus \{x, y\}$ such that $x \; R_i \; z \; R_i \; y$ or $y \; R_i \; z \; R_i \; x$.

Preferences R_i and R_i' are *adjacent* if the only difference between them is the ranks of one pair of adjacent alternatives. Thus, the preferences adjacent to R_i can be obtained by exchanging the positions of one pair of adjacent alternatives in R_i.

A rule f satisfies

(i) *strategy-proofness* if for each $\boldsymbol{R} \in \mathcal{L}^N$, each $i \in N$, and each $R_i' \in \mathcal{L}$,

$$f(\boldsymbol{R}) \; R_i \; f(R_i', \boldsymbol{R}_{-i}).$$

(ii) *monotonicity* if for each $\boldsymbol{R} \in \mathcal{L}^N$, each $i \in N$, and each $R_i' \in \mathcal{L}$ such that $\{x \in X \mid f(\boldsymbol{R}) \; R_i \; x\} \subset \{x \in X \mid f(\boldsymbol{R}) \; R_i' \; x\}$,

$$f(R_i', \boldsymbol{R}_{-i}) = f(\boldsymbol{R}).$$

(iii) *weak monotonicity* if for each $\boldsymbol{R} \in \mathcal{L}^N$, each $i \in N$, and each $R_i' \in \mathcal{L}$ such that $f(\boldsymbol{R})$ raises its position in the passage from R_i to R_i',

$$f(R_i', \boldsymbol{R}_{-i}) = f(\boldsymbol{R}).$$

(iv) *individual bounded response* if for each $\boldsymbol{R} \in \mathcal{L}^N$, each $i \in N$, and each $R_i' \in \mathcal{L}$ which is adjacent to R_i,

$$|\rho_{R_i}(f(\boldsymbol{R}_{-i})) - \rho_{R_i'}(f(R_i', \boldsymbol{R}_{-i}))| \leq 1.$$

(v) *top-restricted strategy-proofness* if for each $\boldsymbol{R} \in \mathcal{L}^N$, each $i \in N$, and each $R_i' \in \mathcal{L}$, it is impossible that $f(\boldsymbol{R})$ is the second preferred and $f(R_i', \boldsymbol{R}_{-i})$ is the most preferred alternative in R_i.

(vi) *top-restricted AM-proofness* if for each $\boldsymbol{R} \in \mathcal{L}^N$, each $i \in N$, and each $R_i' \in \mathcal{L}$ which is adjacent to R_i, it is impossible that $f(\boldsymbol{R})$ is the second preferred and $f(R_i', \boldsymbol{R}_{-i})$ is the most preferred alternative in R_i.[4]

[4]"AM" stands for Adjacent Manipulation (Sato, 2013).

Strategy-proofness ensures that reporting the true preference is always the optimal strategy regardless of what the other agents report. *Monotonicity* says that expanding the lower contour set of the social choice does not change the social choice. This property is known as a necessary condition for *strategy-proofness*, and extensively used in arguments concerning *strategy-proofness*. Moreover, Muller and Satterthwaite (1977) show that on $\mathcal{L}$, *monotonicity* is equivalent to *strategy-proofness*. *Weak monotonicity* considers lifting the position of $f(\boldsymbol{R})$ without any other change in preferences, and requires that the social choice remains the same after the change of preferences. This reasonable requirement is so weak that most "usual" rules satisfy it.[5]

To the best of my knowledge, the last three axioms, *individual bounded response*, *top-restricted strategy-proofness*, and *top-restricted AM-proofness*, are new. We do not think that they are appealing very much by themselves. However, as we present in the next section, they play an important role in understanding *strategy-proofness*.

Individual bounded response says that the smallest change of agent i's preferences leads to the smallest, if any, change of the ranks of the social choice in agent i's preferences. Note that *individual bounded response* means a smooth change of the ranks of the social choice only for the agent who changes preferences, and the ranks of the social choice might change very much for the other agents. On the other hand, *bounded response* of social welfare functions by Sato (2014) and Muto and Sato (2014) means the smallest change, if any, of the social preference to all agents, and these two axioms are logically independent from each other.

Top-restricted strategy-proofness and *Top-restricted AM-proofness* puts a restriction on the working range of *strategy-proofness*. *Top-restricted strategy-proofness* excludes the possibility of changing the social choice from the second to the first

[5]The exception is a plurality with a runoff.

ranked alternative by reporting a false preference, and *top-restricted AM-proofness* restricts the options for misrepresentation to the preferences to the sincere one in *top-restricted strategy-proofness*. Sato (2013) shows that *strategy-proofness* is logically equivalent to *AM-proofness* which restricts the options for misrepresentation to the preferences to the sincere one in *strategy-proofness*. However, as we will see in Section 3.4, *top-restricted strategy-proofness* and *top-restricted AM-proofnes* are not equivalent.

3 Results

In Section 3.1, we present a main result and its proof. Several remarks on the result are in Section 3.2. We see the independence of the axioms in Section 3.3. In Section 3.4, by applying our main theorem, we propose new criteria of nonmanipulability, and find rules satisfying them.

3.1 A theorem

Theorem 3.1

The following statements are equivalent:

(i) A rule satisfies strategy-proofness.

(ii) A rule satisfies top-restricted AM-proofness, weak monotonicity, *and* individual bounded response.

By Muller and Satterthwaite (1977), we can use *monotonicity* instead of *strategy-proofness* in statement (i).

Proof. (i) $\Rightarrow$ (ii). Easy.

(ii) $\Rightarrow$ (i). Let f be a rule satisfying *top-restricted AM-proofness*, *weak monotonicity* and *individual bounded response*. Let $\boldsymbol{R} \in \mathcal{L}^N$ and $i \in N$. Let $R_i' \in \mathcal{L}$

R_i	R_i'	R_i^1	R_i^2	R_i^3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
z	z	$[y]$	y	$[z]$
y	$[y]$	z	$[z]$	y
$[x]$	v	v	x	x
v	x	x	v	v
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Preferences and social choices in the proof of Claim 1.

be a preference adjacent to R_i, and w, v be the alternatives whose ranks are exchanged in the passage from R_i to R_i'. Assume $w\ R_i\ v$. In the following, we prove $f(\boldsymbol{R})\ R_i\ f(R_i', \boldsymbol{R}_{-i})$.

Let $x = f(\boldsymbol{R})$ and $y = f(R_i', \boldsymbol{R}_{-i})$. If $x = y$, $x\ R_i\ y$ by reflexivity of R_i. Assume $x \neq y$. Suppose that $y\ R_i\ x$.

CLAIM 1: $\{x, y\} \cap \{w, v\} = \emptyset$.

Suppose $x = w$. Because $y\ R_i\ x$ and $w\ R_i\ v$, v cannot be y. Because the social choice changes from x to y in the passage from R_i to R_i', by *individual bounded response*, y should be just above x at R_i, and R_i' should be such that $y\ R_i'\ v\ R_i'\ x$. By *top-restricted AM-proofness*, there is $z \in X \setminus \{y\}$ such that $z\ R_i\ y$. This situation is summarized in the first two columns of Table 1. The alternatives within brackets are social choices corresponding to agent i's preferences. By *weak monotonicity*, $f(R_i^1, \boldsymbol{R}_{-i}) = y$. Because R_i^1 and R_i^2 are adjacent, by *individual bounded response*, the candidates for $f(R_i^2, \boldsymbol{R}_{-i})$ are y, z, and the alternative just above y at R_i^2. Because R_i and R_i^2 are adjacent, by *individual bounded response*, the candidates for $f(R_i^2, \boldsymbol{R}_{-i})$ are z, x, and v. Because z is the only alternative in the intersection of these sets of candidates, $f(R_i^2, \boldsymbol{R}_{-i}) = z$. By *weak monotonicity*,

R_i	R'_i	R^1_i	R^2_i	R^3_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
z	z	$[y]$	y	$[z]$
y	$[y]$	z	$[z]$	y
$[x]$	x	x	x	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w	v	v	w	w
v	w	w	v	v
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2: Preferences and social choices in the proof of Claim 2.

$f(R_i^3, \boldsymbol{R}_{-i}) = z$. Because $R_i = R_i^3$, this is a contradiction.

Suppose $x = v$. By *weak monotonicity*, in the passage from R_i to R'_i, the social choice does not change. This is a contradiction.

Suppose $y = w$. By *weak monotonicity*, in the passage from R'_i to R_i, the social choice does not change. This is a contradiction.

Suppose $y = v$. Because $y \ R_i \ x$ and $w \ R_i \ v$, w cannot be x. Thus, R_i and R'_i are such that $w \ R_i \ y \ R_i \ [x]$ and $[y] \ R'_i \ w \ R'_i \ x$, and $\rho_{R_i}(w) = \rho_{R'_i}(y)$. In the passage from R_i from R'_i, the social choice changes from x to y, which is a contradiction to *individual bounded response*.

CLAIM 2: **We have a contradiction in the case where $\{x, y\}$ is above $\{w, v\}$ in R_i.**

This case is described by the first two columns of Table 2. The existence of such $z \in X \setminus \{x, y, w, v\}$ is ensured by *top-restricted AM-proofness*. Then, by *weak monotonicity*, $f(R_i^1, \boldsymbol{R}_{-i})$ should be y. In the passage from R_i^1 to R_i^2, the positions of w and v are exchanged. By *individual bounded response*, the candi-

R_i	R'_i	R^1_i	R^2_i	R^3_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w	v	$[y]$	$[y]$	$[w]$
v	w	v	$[w]$	$[y]$
$\vdots$	$\vdots$	w	v	v
y	$[y]$	$\vdots$	$\vdots$	$\vdots$
$[x]$	x	x	x	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3: Preferences and social choices in the proof of Claim 3.

dates for $f(R_i^2, \boldsymbol{R}_{-i})$ are y, z, and the alternative just above y. Because R_i and R_i^2 are adjacent, by *individual bounded response*, $f(R_i^2, \boldsymbol{R}_{-i}) = z$. Then, by *weak monotonicity*, $f(R_i^3, \boldsymbol{R}_{-i})$ should be z. Because $R_i = R_i^3$, this is a contradiction.

CLAIM 3: **We have a contradiction in the case where $\{x, y\}$ is below $\{w, v\}$ in R_i.**

This case is described by the first two columns of Table 3. Let α be the number of alternatives between v and y in R_i. (It is possible that $\alpha = 0$.) From R'_i, raise the position of y one at a time until we have R_i^1. From R'_i to R_i^1, y overtakes $(\alpha + 2)$ alternatives. By *weak monotonicity*, $f(R_i^1, \boldsymbol{R}_{-i}) = y$. By exchanging the positions of v and w in R_i^1, we have R_i^2. By *individual bounded response*, the candidates for $f(R_i^2, \boldsymbol{R}_{-i})$ are y, w, and the alternative just above y in R_i^2. For convenience, y and w are in the brackets at R_i^2 in Table 3. From R_i^2, lower the position of y one at a time until we have R_i. In this procedure, R_i^3 is the first preference and R_i is the $(\alpha + 2)$th preference from R_i^2. By *weak monotonicity* and *individual bounded response*, the candidates for $f(R_i^3, \boldsymbol{R}_{-i})$ are w, y, and the alternative just above w at R_i^3. In each case, by *individual bounded response*, $f(\boldsymbol{R})$ cannot be x, which is a contradiction. (From R_i^3, for the social choice to be x at R_i, it should move down at least $(\alpha + 2)$ ranks, but from R_i^3 to R_i, there are only $(\alpha + 1)$ preferences.)

3.2 Discussion

On contagion of nonmanipulability. In the Introduction, we discussed our result by setting *weak monotonicity* as the starting point of looking for desirable rules. Here, let us set *top-restricted AM-proofness* as the starting point. Then, our result says that by additionally imposing *weak monotonicity* and *individual bounded response*, we reach *strategy-proofness*. Remember that *Top-restricted AM-proofness* assumes "local" nonmanipulability in the sense that it considers only the change of social choice from the second to the most preferred alternative. Our result implies that *weak monotonicity* and *individual bounded response* spread the "local" nonmanipulability to the "global" nonmanipulability, i.e., *strategy-proofness*.

On *top-restricted AM-proofness*. We could use an alternative axiom instead of *top-restricted AM-proofness* in Theorem 3.1. For example, instead of a false preference which is adjacent to the sincere one, a false preference could be any element of $\mathcal{L}$, i.e., it could be *top-restricted strategy-proofness*. Also, instead of excluding the possibility of achieving the most preferred alternative from the second one, we could exclude the possibility of achieving the most preferred alternative from any other one. We employ *top-restricted AM-proofness* because it seems to be the weakest one among the axioms leading to the equivalence in Thoerem 3.1.

On domain restrictions. Our theorem is about the rules on the universal domain of preferences. A natural question is whether the theorem holds on restricted domains. Unfortunately, we do not have a clear answer to this question. However, at some specific cases, we can see that the equivalence does not hold. For example, consider a minimal circular domain (Sato, 2010) with $|X| \geq 4$. Then, because no distinct preferences are adjacent, *top-restricted AM-proofness* and *individual bounded response* lose their power completely. Thus, each rule trivially satisfies them. Therefore, the plurality rule satisfies *top-restricted AM-proofness*, *weak monotonicity*, and *individual bounded response*, whereas it violates *strategy-proofness*.

3.3 Independence of axioms

Our result characterizes *strategy-proofness* in terms of three axioms, *top-restricted AM-proofness*, *weak monotonicity*, and *individual bounded response*. In the following, we show that for each selection of two axioms from the three, there is a rule which satisfies them and violates the remaining one.

Only *top-restricted AM-proofness* and *weak monotonicity*. A plurality rule with an appropriate tie-breaking rule satisfies *top-restricted AM-proofness* and *weak monotonicity* and violates *individual bounded response*. Because a plurality rule also satisfies *top-restricted strategy-proofness*, we can repeat this argument with *top-restricted strategy-proofness* instead of *top-restricted AM-proofness*.

Only *top-restricted AM-proofness* and *individual bounded response*. Choose $i \in N$, and consider the rule choosing the worst alternative according to R_i. This rule satisfies *top-restricted AM-proofness* and *individual bounded response* and violates *weak monotonicity*.

Only *weak monotonicity* and *individual bounded response*. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $i \in N$. Define the rule f by for each $\boldsymbol{R} \in \mathcal{L}^N$,

$$f(\boldsymbol{R}) = \begin{cases} x_2, & \text{if } x_1 \; R_i \; x_2 \; R_i \; x_3 \; R_i \; x_4 \ldots R_i \; x_m, \\ \text{the most preferred alternative at } R_i, & \text{otherwise.} \end{cases}$$

It can be seen that f satisfies *weak monotonicity* and *individual bounded response*, but violates *top-restricted AM-proofness*.

3.4 A useful nonmanipulability condition

Theorem 3.1 decomposes *strategy-proofness* into *top-restricted AM-proofness*, *weak monotonicity*, and *individual bounded response*. Among them, we believe that *individual bounded response* is not as defensible as the other two. Then, what is the rules satisfying *top-restricted AM-proofness* and *weak monotonicity*? We have a partial answer to this question.

A rule f is a *scoring rule* if there are real numbers $s_1, s_2, \ldots, s_m$ such that

(i) $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$, and

(ii) $f(\boldsymbol{R}) \in \{x \in X \mid \sum_{i \in N} s_{\rho_{R_i}(x)} \geq \sum_{i \in N} s_{\rho_{R_i}(y)} \text{ for each } y \in X\}$.

For each k $(1 \leq k \leq m)$, s_k is the score of the kth ranked alternative. Given $\boldsymbol{R} \in \mathcal{L}^N$, $\rho_{R_i}(x)$ is the rank of x at R_i. Thus, $s_{\rho_{R_i}(x)}$ is the score of x at R_i, and $\sum_{i \in N} s_{\rho_{R_i}(x)}$ is the total score of x. A *scoring rule* selects an alternative with the highest score. In the following, let $X = \{x_1, x_2, \ldots, x_m\}$ and we assume that $f(\boldsymbol{R})$ is the alternative with the smallest index among the alternatives with the highest score.

It is easy to see that scoring rules satisfy *weak monotonicity*. We give two results which show how *top-restricted AM-proofness* and *top-restricted strategy-proofness* narrow down the class of scoring rules, respectively.

Proposition 3.1

For each scoring rule f, the following statements are equivalent:

(i) f satisfies top-restricted AM-proofness.

(ii) $s_2 = s_3$.

α	α	1
x_1	x_2	x_3
x_2	x_1	x_2
x_3	x_3	x_1
$\vdots$	$\vdots$	$\vdots$

Table 4: Preferences for odd n in the proof of Proposition 3.1.

Proof. (i) $\Rightarrow$ (ii). Let f be a scoring rule satisfying *top-restricted AM-proofness*. Suppose $s_2 > s_3$.

Case 1: n is an even number. Let $\boldsymbol{R} \in \mathcal{L}^N$ be such that for each i $(1 \leq i \leq \frac{n}{2})$, $x_1\ R_i\ x_2\ R_i\ x_3\ \ldots\ R_i\ x_m$, and each i' $(\frac{n}{2}+1 \leq i' \leq n)$, $x_2\ R_{i'}\ x_1\ R_{i'}\ x_3\ \ldots\ R_{i'}\ x_m$. Then, x_1 and x_2 attain the highest score and $f(\boldsymbol{R}) = x_1$. Consider that agent n changes his preference from R_n to R'_n such that $x_2\ R'_n\ x_3\ R'_n\ x_1\ \ldots\ R'_n\ x_m$. Note that R_n and R'_n are adjacent. Then, the score of x_1 decreases and x_2 attains the highest score. (The score of x_3 increases, but all agents rank x_2 above x_3, and agent 1 ranks them at the second and the third position, and hence x_3's score is lower than x_2's score.) Therefore, agent n can change the social choice from the second one to the most preferred one by reporting a false preference which is adjacent to the sincere one.

Case 2: n is an odd number. Let $\alpha = \frac{n-1}{2}$. Let $\boldsymbol{R} \in \mathcal{L}^N$ be such that for each i $(1 \leq i \leq \frac{n-1}{2})$, $r^1(R_i) = x_1\ R_i\ x_2\ R_i\ x_3 = r^3(R_i)$, each i' $(\frac{n-1}{2} + 1 \leq i' \leq n-1)$, $r^1(R_{i'}) = x_2\ R_{i'}\ x_1\ R_{i'}\ x_3 = r^3(R_{i'})$, and $r^1(R_n) = x_3\ R_n\ x_2\ R_n\ x_1 = r^3(R_n)$, and the unspecified parts can be arbitrary. The situation is summarized in Table 4. The first row of Table 4 shows the numbers of agents having the preference below the numbers. The scores at $\boldsymbol{R}$ are

- x_1: $\alpha s_1 + \alpha s_2 + s_3$,

- x_2: $\alpha s_1 + \alpha s_2 + s_2$,
- x_3: $s_1 + 2\alpha s_3$,

and the score of each other alternative, if any, is not greater than these scores. Since $s_2 > s_3$, $f(\boldsymbol{R}) = x_2$. Let $R'_1 \in \mathcal{L}$ be such that $r_1(R'_1) = x_1 \; R'_1 \; x_3 \; R'_1 \; x_2$, and the other parts are the same as in R_1. Then, R_1 and R'_1 are adjacent, and the scores at $(R'_1, \boldsymbol{R}_{-1})$ are

- x_1: $\alpha s_1 + \alpha s_2 + s_3$,
- x_2: $\alpha s_1 + \alpha s_2 + s_3$,
- x_3: $s_1 + s_2 + (2\alpha - 1)s_3$,

and the score of each other alternative, if any, is not greater than these scores. Thus, the scores of x_1 and x_2 are the same, and by induction on α, it can be seen that the score of x_3 is not greater than x_1's score. Therefore, $f(R'_1, \boldsymbol{R}_{-1}) = x_1 = r^1(R_1)$. This is a contradiction to *top-restricted AM-proofness*.

(ii) $\Rightarrow$ (i). Assume $s_2 = s_3$. Let $\boldsymbol{R} \in \mathcal{L}^N$ and $i \in N$. Assume that $f(\boldsymbol{R})$ is the second preferred alternative according to R_i. To change the social choice from $f(\boldsymbol{R})$ to the most preferred alternative according to R_i, it is necessary to increase the score of the most preferred alternative or decrease the score of $f(\boldsymbol{R})$. However, by reporting a false preference which is adjacent to the sincere one, it is impossible to achieve this change. $\square$

The following result says that among the scoring rules, the plurality is the only one satisfying *top-restricted strategy-proofness*.

Proposition 3.2

Let f be a scoring rule. When n is even, the following statements are equivalent:

(i) f satisfies top-restricted strategy-proofness.

(ii) $s_1 > s_2 = s_3 = \cdots = s_m$, *i.e., f is the plurality rule.*

Proof. (i) ⇒ (ii). Let f be a scoring rule satisfying *top-restricted strategy-proofness*.

CLAIM 1: $s_1 > s_2$.

Suppose $s_1 = s_2$. Let $k \geq 2$ be the smallest integer such that $s_k > s_{k+1}$. Let $\boldsymbol{R} \in \mathcal{L}^N$ be such that for each $i \in N$, $x_2 \; R_i \; x_1 \; R_i \; x_3 \; \ldots \; R_i \; x_m$. Then, x_1, $x_2, \ldots, x_k$ attain the highest score, and $f(\boldsymbol{R}) = x_1$. At R_1, lower the position of x_1 to the $(k+1)$th rank. This decreases the score of x_1, and $x_2, \ldots, x_k$ attain the highest score, and a new social choice is x_2. This is a contradiction to *top-restricted strategy-proofness*.

CLAIM 2: $s_2 = s_3 = \cdots = s_m$.

Assume that $s_k > s_{k+1}$ for some k $(2 \leq k \leq m-1)$. Especially, let $k \geq 2$ be the smallest integer such that $s_k > s_{k+1}$.

Case 1: n is an even number. Let $\boldsymbol{R} \in \mathcal{L}^N$ be such that for each i $(1 \leq i \leq \frac{n}{2})$, $x_1 \, R_i \, x_2 \, R_i \, x_3 \; \ldots \; R_i \, x_m$, and each i' $(\frac{n}{2}+1 \leq i' \leq n)$, $x_2 \, R_{i'} \, x_1 \, R_{i'} \, x_3 \; \ldots \; R_{i'} \, x_m$. Then, x_1 and x_2 attain the highest score and $f(\boldsymbol{R}) = x_1$. At R_n, lower the position of x_1 to the $(k+1)$th rank. This decreases the score of x_1, and x_2 attains the highest score, and a new social choice is x_2. (All agents rank x_2 above each alternative except x_1, and some agent ranks x_2 at the top. Thus, the score of x_2 is strictly larger than the other alternatives.) This is a contradiction to *top-restricted strategy-proofness*.

Case 2: n is an odd number. Let $\alpha = \frac{n-1}{2}$. Let $\boldsymbol{R} \in \mathcal{L}^N$ be such that for each i $(1 \leq i \leq \frac{n-1}{2})$, $r^1(R_i) = x_1 \; R_i \; x_2 \; R_i \; x_3 = r^3(R_i)$ and $r^m(R_i) = x_m$, each i' $(\frac{n-1}{2} + 1 \leq i' \leq n-1)$, $r^1(R_{i'}) = x_2 \; R_{i'} \; x_1 \; R_{i'} \; x_m = r^3(R_{i'})$ and $r^m(R_{i'}) = x_3$, and $r^1(R_n) = x_3 \; R_n \; x_2 \; R_n \; x_1 = r^3(R_n)$, and the unspecified parts are arbitrary. The situation is summarized in Table 5. The scores at $\boldsymbol{R}$ are

α	α	1
x_1	x_2	x_3
x_2	x_1	x_2
x_3	x_m	x_1
$\vdots$	$\vdots$	$\vdots$
x_m	x_3	

Table 5: Preferences for odd n in the proof of Proposition 3.2.

- x_1: $\alpha s_1 + \alpha s_2 + s_3$
- x_2: $\alpha s_1 + (\alpha + 1)s_2$
- x_3: $s_1 + \alpha s_3 + \alpha s_m$

and the score of each other alternative is less than x_2's score. If $s_2 = s_3$, i.e., $k > 2$, then $f(\boldsymbol{R}) = x_1$. If $s_2 > s_3$, then $f(\boldsymbol{R}) = x_2$.

Subcase 2.1: $f(\boldsymbol{R}) = x_1$**.** At R_{n-1}, lower the position of x_1 to the bottom. Then, the scores at $(R'_{n-1}, \boldsymbol{R}_{n-1})$ are

- x_1: $\alpha s_1 + (\alpha - 1)s_2 + s_3 + s_m$
- x_2: $\alpha s_1 + (\alpha + 1)s_2$
- x_3: $s_1 + \alpha s_3 + s_{m-1} + (\alpha - 1)s_m$

and the score of each other alternative is less than x_2's score. Since $s_2 > s_m$, x_2's score is larger than x_1's score. Thus, $f(R'_{n-1}, \boldsymbol{R}_{n-1}) = x_2$. This is a contradiction to *top-restricted strategy-proofness*.

Subcase 2.2: $f(\boldsymbol{R}) = x_2$**.** At R_1, lower the position of x_2 to the bottom. Then, the scores at $(R'_1, \boldsymbol{R}_{-i})$ are

- x_1: $\alpha s_1 + \alpha s_2 + s_3$

- x_2: $\alpha s_1 + \alpha s_2 + s_m$
- x_3: $s_1 + s_2 + (\alpha - 1)s_3 + \alpha s_m$

and the score of each other alternative is less than x_1's score, and we have $f(R'_1, \boldsymbol{R}_{-1}) = x_1$. This is a contradiction to *top-restricted strategy-proofness*.

(ii) $\Rightarrow$ (i). Easy. $\square$

The above two results shed new light on the study of nonmanipulability of rules. Especially, Proposition 3.2 says that we can recommend a plurality rule as the only scoring rule satisfying *top-restricted strategy-proofness*. We know many impossibility theorems on *nonmanipulability*. They are important in finding what we cannot do. However, they rarely have practical implications, while our results have. Remember that *top-restricted strategy-proofness* is not an ad hoc axiom. When *individual bounded response* is removed from *strategy-proofness*, we are left with *weak monotonicity* and *top-restricted strategy-proofness*.

Finally, we give an equivalent condition to *top-restricted strategy-proofness* under *weak monotonicity*.

Proposition 3.3

Let f be a rule satisfying weak monotonicity. *Then, the following statements are equivalent:*

(i) f *satisfies* top-restricted strategy-proofness.

(ii) For each $\boldsymbol{R} \in \mathcal{L}^N$ and each $R'_i \in \mathcal{L}$,

$$f(\boldsymbol{R}) = r^1(R_i) = r^1(R'_i) \Rightarrow f(R'_i, \boldsymbol{R}_{-i}) = f(\boldsymbol{R}).$$

Proof. (i) $\Rightarrow$ (ii). Let f be a rule satisfying *top-restricted strategy-proofness*. Let $\boldsymbol{R} \in \mathcal{L}^N$, $i \in N$, and $R'_i \in \mathcal{L}$ be such that $f(\boldsymbol{R}) = r^1(R_i) = r^1(R'_i)$. Suppose

$f(R'_i, \boldsymbol{R}_{-i}) \neq f(\boldsymbol{R})$. Since $f(\boldsymbol{R}) = r^1(R'_i)$, we have $f(R'_i, \boldsymbol{R}_{-i} = r^k(R'_i)$ for some $k \geq 2$. Let $R''_i \in \mathcal{L}$ be such that $f(R'_i, \boldsymbol{R}_{-i})$ raises its position from the kth to the second position in the passage from R'_i to R''_i. By *weak monotonicity*, $f(R''_i, \boldsymbol{R}_{-i}) = f(R'_i, \boldsymbol{R}_{-i})$. According to R''_i, $f(R''_i, \boldsymbol{R}_{-i})$ is the second preferred, and $f(\boldsymbol{R})$ is the most preferred alternative. This is a contradiction to *top-restricted strategy-proofness*.

(ii) $\Rightarrow$ (i). Assume (ii). Suppose that f violates *top-restricted strategy-proofness*, i.e., $r^1(R_i) = f(R'_i, \boldsymbol{R}_{-i})$ and $r^2(R_i) = f(\boldsymbol{R})$ for some $\boldsymbol{R} \in \mathcal{L}^N$, $i \in N$, and $R'_i \in \mathcal{L}$. Let $R''_i \in \mathcal{L}$ be such that $f(R'_i, \boldsymbol{R}_{-i})$ raises its position to the top in the passage from R'_i to R''_i. By *weak monotonicity*, $f(R''_i, \boldsymbol{R}_{-i}) = f(R'_i, \boldsymbol{R}_{-i})$. Then, we have $r^1(R''_i) = f(R''_i, \boldsymbol{R}_{-i}) = f(R'_i, \boldsymbol{R}_{-i}) = r^1(R_i)$, and $f(R''_i, \boldsymbol{R}_{-i}) \neq f(\boldsymbol{R})$. This is a contradiction to (ii). $\square$

4 Concluding remarks

Form the broad viewpoint, this research is a part of investigating the possibility of constructing a nonmanipulable rule. We succeed to some extent.

It is well known that when we adopt *strategy-proofness* as a nonmanipulability, we end up with impossibility in most cases. Thus, weaker notions of nonmanipulability than *strategy-proofness* are called for. By decomposing *strategy-proofness* into three independent axioms, we have options for such weaker notions. Our impression is that *individual bounded response* is not as defensible as the other two. Thus, the next step is a characterization of rules satisfying [*weak monotonicity* and *top-restricted AM-proofness*] or [*weak monotonicity* and *top-restricted strategy-proofness*], and some other mild conditions. We show that in the class of scoring rules, the plurality rule is the only one satisfying *top-restricted strategy-proofness*. Considering that the plurality rule is often criticized by its "poor" performance, this is an interesting result supporting the plurality rule.

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