Interpersonal Comparison Necessary for Arrovian Aggregation

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Abstract

While studies of social welfare functionals have revealed that some interpersonal comparability, such as ordinal and level comparability and cardinal and unit comparability, resolves Arrow's impossibility theorem, we have not known yet what kind of information is necessary to resolve it. The purpose of this paper is to capture the feature of informational structures which make social welfare functionals satisfying *Strong Pareto*, *Anonymity* and *Independence of Irrelevant Alternatives* available. To do so, we provide a characterization of such informational structures. We know from this characterization that if utility levels are not interpersonally comparable, then transformed utility functions by a certain transformation need to be cardinal and unit comparable.

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1 Introduction

We consider the problem of an ethical observer who wants to aggregate individuals' real-valued utility functions into an ordering. Sen [27] first defined such an aggregation procedure as a *social welfare functional* (SWFL), which is a generalization of Arrow's *social welfare function* (Arrow [2].) Studies of SWFLs have revealed that some interpersonal comparability, such as *ordinal and level comparability* (OLC) and *cardinal and unit comparability* (CUC), makes it possible to aggregate individuals' utility functions through SWFLs satisfying *Strong Pareto*, *Anonymity* and *Independence of Irrelevant Alternatives*, contrary to Arrow's impossibility theorem[1]. If utility functions are ordinal and level comparable, which allows interpersonal comparisons of utility *level*s, then the leximin rule are available[2]. If utility functions are cardinal and unit comparable, which allows interpersonal comparisons of utility *differences*, then the utilitarian rule is available[3]. Thus, in order resolve Arrow's impossibility theorem, it is *sufficient* to permit such interpersonal comparisons.

On the other hand, we have not known yet whether some interpersonal comparisons are *necessary* to resolve it. As is noted by d'Aspremont and Gevers [12], previous studies have treated some specific informational structures. They have not checked whether any given informational structure provides a way of escape route from Arrow's impossibility theorem or not. The purpose of this paper is to capture the feature of informational structures with which *Strong Pareto*, *Anonymity* and *Independence of Irrelevant Alternatives* are consistent. To do so, we provide a characterization of such informational structures.

[1] For surveys of this literature, see Blackorby, Bossert and Donaldson [4], Blackorby, Donaldson, and Weymark [5], Bossert and Weymark [8], d'Aspremont [9], d'Aspremont and Gevers [11], Fleaurbaey and Hammond [14], Gartner [15] Roberts [25], Roemer [26], and Sen [29] [30].

[2] For this and relatd results, see d'Aspremont and Gevers [12], Hammond [17] [18], Gevers [16], Deschamps and Gevers [12] Roberts [23] and Sen [28] [29].

[3] For this and related results, see Basu [3], d'Aspremont and Gevers [10], Deschamps and Gevers [12], Gevers [16], Maskin [19], and Roberts [24].

Our approach is summarized as follows. First, we describe an *informational structure* of utility functions as an equivalence relation over the set of utility profiles[4]. An equivalence relation determines which pairs of utility profiles are informationally equivalent. For each equivalence relation, we define an information invariance axiom, which demands that if two utility profiles are regarded as informationally equivalent, then they should be treated in the same way. We study all feasible equivalence relations which are independent of the names of individuals and can be defined in terms of sets of *invariance transformations* which do not distort the information utility functions convey.

Second, we assume that utility *levels* are not interpersonally comparable. This is because we already know that interpersonal *level* comparability makes it possible to use the leximin rule, which satisfies *Strong Pareto*, *Anonymity* and *Independence of Irrelevant Alternatives*. We conversely want to know what kind of utility information is necessary without interpersonal *level* comparisons. Deschamps and Gevers [12] note that if interpersonal *level* comparisons are prohibited, then changes of utility *levels* of those who are indifferent among all alternatives are to be ignored. This requirement is called *Separability*. We impose *Separability* as an assumption on informational structures.

Third, within such invariance axioms, we characterize the class of invariance axioms which are compatible with *Strong Pareto*, *Anonymity* and *Independence of Irrelevant Alternatives* (Theorem 3). This theorem says that if utility *levels* are not interpersonally comparable, then transformed utilities functions by a certain transformation need to be cardinal and unit comparable. That is, there is a transformation g with which the ethical observer can treat transformed utility functions $(g \circ U_1, \ldots, g \circ U_n)$ as cardinal and unit comparable. That is, *differences* of transformed utilities by g are interpersonally comparable.

It is well known that cardinal and unit comparability is closely related to the utilitarian rule. Similarly with this relationship, cardinal and unit comparability of transformed utilities by g is also closely related to a *transformed utilitarian rule associated with* g, which compares each pair of alternatives through the sum of transformed utilities by g. We show this fact by providing a characterization of a transformed utilitarian rule based on cardinal and unit comparability of trans-

[4] Such a description of information structures is proposed by Bossert [6] [7] and Fleurbaey [13].

formed utilities by g (Proposition 5).

The rest of this paper is organized as follows: Section 2 introduces some definitions; Section 3 defines several invariance axioms; Section 4 establishes our main results; and Section 5 provides a conclusion. Parts of proofs are relegated to the Appendix.

2 Basic Definitions

Let $N \equiv \{1, 2, \cdots, n\}$ be the set of *individuals* and X be the set of *alternatives.* We suppose that $2 \leq |N| < \infty$ and $|X| \geq 3$, where $|A|$ denotes the cardinality of the set A. Let $V \subseteq \mathbb{R}$ denote the set of *utility values.* We suppose that V is *order isomorphic*[5] to $\mathbb{R}$. Let $\mathcal{G} \equiv \{g : V \rightarrow \mathbb{R} : g \textit{ is increasing and onto}\}$ be the class of order isomorphic transformations.

Given V, for each $i \in N$, *individual i's utility function* is a mapping $U_i : X \rightarrow V$ which associates with each alternative $x \in X$ a utility value $U_i(x) \in V$. Let $\mathcal{U}$ be the set of possible utility functions. A *utility profile* is an n-tuple $U = (U_1, \cdots, U_n)$, where for each $i \in N$, $U_i \in \mathcal{U}$. The set of utility profiles is denoted by $\mathcal{U}^N \equiv \prod_{i \in N} \mathcal{U}$. Let $\mathcal{R}$ be the set of social preference relations over X which are *reflexive*, *complete* and *transitive.*

A *social welfare functional* (SWFL) is a mapping $R : \mathcal{U}^N \rightarrow \mathcal{R}$ which associates with each utility profile $U \in \mathcal{V}^N$ a social preference relation $R(U) \in \mathcal{R}$. We suppose that the domain of a SWFL is unrestricted. To simplify notation, let $R_U \equiv R(U)$. The symmetric and asymmetric parts of R_U are denoted by I_U and P_U, respectively. A *social welfare ordering* (SWO) is a binary relation $\succsim$ over $V^N = \prod_{i \in N} V$ which are *reflexive*, *complete* and *transitive.* The symmetric and asymmetric parts of $\succsim$ are denoted by $\succ$ and $\sim$, respectively.

We require SWFLs to satisfy the following three well-known axioms. First, (1) if all individuals' utility values at x are least as high as at y, then x should be at least as desirable as y and (2) if additionally there is an individual whose utility value at x is higher than y, then x should be socially preferred to y.

[5] V is *order isomorphic* to $\mathbb{R}$ if and only if there is an increasing and onto *transformation* $g : V \rightarrow \mathbb{R}$. For example, since $\log : \mathbb{R}_{++} \rightarrow \mathbb{R}$ is increasing and outo, $\mathbb{R}_{++}$ is order isomorphic to $\mathbb{R}$.

Strong Pareto (SP): *For each pair* $x, y \in X$, *and each* $U \in \mathcal{U}^N$, *(1) if for each* $i \in N$, $U_i(x) \geq U_i(y)$, *then* $x \, R_U \, y$, *and (2) if for each* $i \in N$, $U_i(x) \geq U_i(y)$, *and for some* $j \in N$, $U_j(x) > U_j(y)$, *then* $x \, P_U \, y$.

Second, a social preference should not depend on the names of individuals.

Anonymity (AN): *For each permutation* π *of* N, $R(U) = R(\pi(U))$.

Third, the social ranking of any two alternatives should be independent of the utilities derived from the other alternatives.

Independence of Irrelevant Alternatives (IIA): *For each pair* $x, y \in X$, *and each pair* $U, U' \in \mathcal{U}^N$, *if for each* $i \in N$, $U_i(x) = U'_i(x)$ *and* $U_i(y) = U'_i(y)$, *then* $x \, R_U \, y \Leftrightarrow x \, R_{U'} \, y$.

We say that a SWFL is *Arrovian*[6] if it satisfies the three axioms above. Our goal is to specify informational structures which make Arrovian SWFLs available. As examples, we present the following SWFLs.

Utilitarian rule: *For each* $U \in \mathcal{U}^N$ *and each pair* $x, y \in X$, $x \, P_U \, y$ *if and only if* $\sum_{i \in N} U_i(x) > \sum_{i \in N} U_i(y)$.

Nash rule: *Suppose that* $V = \mathbb{R}_{++}$. *For each* $U \in \mathcal{U}^N$ *and each pair* $x, y \in X$, $x \, P_U \, y$ *if and only if* $\prod_{i \in N} U_i(x) > \prod_{i \in N} U_i(y)$.

Transformed utilitarian rule: *There is* $g \in \mathcal{G}$ *such that for each* $U \in \mathcal{U}^N$ *and each pair* $x, y \in X$, $x \, P_U \, y$ *if and only if* $\sum_{i \in N} g(U_i(x)) > \sum_{i \in N} g(U_i(y))$.

Leximin rule: *For each* $u \in V^N$ *let* π_u *be a permutation of* N *such that* $u_{\pi_u(1)} \leq$

[6] Arrow [2] considered an aggregation satisfying Weak Pareto, Non Dictatorship and Independence of Irrelevant Alternatives, so these conditions are actually stronger than Arrow's original conditions.

$\ldots \leq u_{\pi_u(n)}$. *For each* $U \in \mathcal{U}^N$ *and each pair* $x, y \in X$, $x\, P_U\, y$ *if and only if there is there is* $j \in \{1, \ldots, n\}$ *such that* $U_{\pi_{U(x)}(j)}(x) > U_{\pi_{U(y)}(j)}(y)$ *and for each* $k \in \{1, \ldots, j-1\}$, $U_{\pi_{U(x)}(k)}(x) = U_{\pi_{U(y)}(k)}(y)$.

We can easily check that any transformed utilitarian rule is Arrovian. Notice that the Nash rule is also a transformed utilitarian rule because $\prod_{i \in N} U_i(x) > \prod_{i \in N} U_i(y)$, if and only if $\sum_{i \in N} \log(U_i(x)) > \sum_{i \in N} \log(U_i(y))$, where $\log : \mathbb{R}_{++} \to \mathbb{R}$ is an increasing and onto transformation.

3 Information Invariance

An *informational structure*[7] specifies which utility profiles are considered to be informationally equivalent from the ethical observer's point of view. Bossert [7] notes that the most general way to describe an informational structure is to define it as an *equivalence relation* [8] over $\mathcal{U}^N$. Given an equivalence relation $\mathcal{I}$, for each pair $U, U' \in \mathcal{U}^N$, $U\, \mathcal{I}\, U'$ means that the ethical observer regards U' as informationally equivalent to U. If two utility profiles U and U' are informationally equivalent, then U and U' should be treated in the same way.

Invariance with respect to $\mathcal{I}$ (INV-$\mathcal{I}$): *For each pair* U,$U' \in \mathcal{U}^N$, *if* $U\, \mathcal{I}\, U'$ *then* $R(U) = R(U')$.

In this paper, we study all informational structures which satisfy the following three assumptions.

First, since we are interested in anonymous aggregation, we require that all individuals' utility functions should be treated in the same way. Then, an equivalence relation should be independent of the names of individuals.

Symmetry (SYM): *For each pair* $U, U' \in \mathcal{U}^N$ *and each permutation* π *of* N,

[7] It is equivalent to what Sen [27] calls an *informational basis*.

[8] An equivalence relation is a binary relation $\mathcal{I}$ which satisfies *reflexivity* $[\forall U \in \mathcal{U}^N,\ U\, \mathcal{I}\, U]$, *symmetry* $[\forall U, U' \in \mathcal{U}^N$, *if* $U\, \mathcal{I}\, U'$, *then* $U'\, \mathcal{I}\, U]$ and *transitivity* $[\forall U, U', U'' \in \mathcal{U}^N$, *if* $U\, \mathcal{I}\, U'$ *and* $U'\, \mathcal{I}\, U''$, *then* $U\, \mathcal{I}\, U'']$.

if $U \mathcal{I} U'$ *then* $(U_{\pi(1)}, \ldots, U_{\pi(n)}) \mathcal{I} (U'_{\pi(1)}, \ldots, U'_{\pi(n)})$.

Second, following the classical informational approach, we assume that an equivalence relation can be defined by means of a set of *invariance transformations* which do not distort the information utility functions convey. Let Ψ be the set of increasing transformations $\varphi : V \to V$ and let $\Psi^N \equiv \prod_{i \in N} \Psi$. For $\Phi^N \subseteq \Psi^N$, we say that Φ^N *represents* $\mathcal{I}$ if for each pair $U, U' \in \mathcal{U}^N$, $U \mathcal{I} U'$ if and only if there is $(\varphi_1, \cdots, \varphi_n) \in \Phi^N$ such that $U' = (\varphi_1 \circ U_1, \cdots, \varphi_n \circ U_n)$. As pointed out by Roberts [24], in order for Φ^N to represent some equivalence relation, Φ^N should be a *subgroup*[9] of Ψ^N with respect to the function composition operator $\circ$.

Representability (REP): There is a subgroup $\Phi^N \subseteq \Psi^N$ such that for each pair $U, U' \in \mathcal{U}^N$, $U \mathcal{I} U'$ if and only if there is $(\varphi_1, \cdots, \varphi_n) \in \Phi^N$ such that $U' = (\varphi_1 \circ U_1, \cdots, \varphi_n \circ U_n)$.

If $\mathcal{I}$ is represented by Φ^N, then INV-$\mathcal{I}$ is equivalent to the following axiom.

Invariance with respect to Φ^N (INV-Φ^N): *For each* $U \in \mathcal{U}^N$ *and each* $(\varphi_1, \cdots, \varphi_n) \in \Phi^N$, $R(U) = R(\varphi_1 \circ U_1, \cdots, \varphi_n \circ U_n)$.

Third, we suppose that utility *levels* are not interpersonally comparable. This is because we already know that interpersonal *level* comparability makes it possible to use Arrovian SWFLs, such as the leximin rule. We conversely want to know what kind of information is necessary for Arrovian aggregation without interpersonal *level* comparisons.

An *unconcerned* individual, defined by Sen [27], is an individual who are indifferent among all alternatives in X. Deschamps and Gevers [12] point out that if the ethical observer cannot make interpersonal comparisons of utility *level*s, then he cannot use the information about utility *levels* of *unconcerned* individuals either. Following Deschamps and Gevers [12], we prohibit interpersonal utility *level*

[9] A subset $\Phi^N \subseteq \Psi^N$ is a *subgroup* of Ψ^N if Φ^N satisfies *reflexivity* $[(id_v, \ldots, id_V) \in \Phi^N$, *where* id_v *denotes the identity mapping of* $V]$, *symmetry* $[\forall \varphi \in \Phi^N,\ \varphi^{-1} \in \Phi^N]$ and *transitivity* $[\forall \varphi, \psi \in \Phi^N,\ \varphi \circ \psi \in \Phi^N]$.

comparisons by requiring that the information about utility *levels* of *unconcerned* individuals is not available.

Separability (SEP): For each pair $U, U' \in \mathcal{U}^N$, $U \mathcal{I} U'$ if there is $S \subseteq N$, such that for each $i \in S$ and each $x \in X$, $U_i(x) = U'_i(x)$, and for each $i \in N \backslash S$ and each pair $x, y \in A$, $U_i(x) = U_i(y)$ and $U'_i(x) = U'_i(y)$.

The only difference between U and U' is the difference of utility *levels* of unconcerned individuals. SEP says that changes of utility *levels* of unconcerned individuals do not matter from the ethical observer's point of view[10]. While previous studies define SEP as an axiom imposed on SWFLs or SWOs, we define SEP as one imposed on informational structures. Since SEP requests a parsimonious attitude toward information, it is also natural to regard SEP as an assumption on informational structures.

It is useful to capture the feature of equivalence relations satisfying SYM, REP and SEP. For $\Phi \subseteq \Psi$, we say that Φ is *rich* (RC) if $id_v \in \Phi$ and for each pair $u, u' \in V$, there is $\varphi \in \Phi$ such that $\varphi(u) = u'$. The following proposition characterizes the class of such equivalence relations.

Proposition 1. *Suppose that $\mathcal{I}$ satisfies* SYM *and* REP. *Let Φ^N represent $\mathcal{I}$. $\mathcal{I}$ satisfies* SEP *if and only if there is rich $\Phi \subseteq \Psi$ such that $\prod_{i \in N} \Phi \subseteq \Phi^N$.*

Proof. See appendix. ■

4 Results

4.1 Cartesian Product Case

The goal of this paper is to specify informational structures which resolve Arrow's impossibility theorem. To do so, we characterize the class of invariance axioms which are compatible with SP, AN and IIA.

[10] Notice that while the prohibition of interpersoal level comparisons implies SEP, SEP does not necessarily imply the prohibition of interpersonal level comparisons, so that SEP is at least as weak as the prohibition of interpersonal level comparison.

We begin our discussion by considering the case where an equivalence relation $\mathcal{I}$ is represented by a Cartesian product of common individual transformations. That is, there is $\Phi \subseteq \Psi$ such that $\prod_{i \in N} \Phi$ represents $\mathcal{I}$. If $\Phi^N = \prod_{i \in N} \Phi$ and Φ, then INV-Φ^N can be rewritten as follows.

Invariance with respect to Φ (INV-Φ): *For each $U \in \mathcal{U}$, each $i \in N$, and each $\varphi \in \Phi$, $R(U) = R(\varphi \circ U_i, U_{-i})$.*

With INV-Φ, Φ indicates the set of *individual invariance transformations* which do not distort the information each individual's utility function conveys. We present some invariance axioms which are defined as INV-Φ.

Invariance with respect to increasing transformations (INV-Φ^{IT}): $\Phi^{IT} = \Psi$.

Invariance with respect to positive affine transformations (INV-Φ^{PAT}): $\Phi^{PAT} = \{\varphi \in \Psi | \ \exists a > 0, \ \exists b \in \mathbb{R} \text{ such that } \forall u \in V, \ \varphi(u) = au + b\}$.

Invariance with respect to origin transformations (INV-Φ^{OT}): $\Phi^{OT} = \{\varphi \in \Psi | \ \exists b \in \mathbb{R} \text{ such that } \forall u \in V, \ \varphi(u) = u + b\}$.

Invariance with respect to scale transformations (INV-Φ^{ST}): $\Phi^{ST} = \{\varphi \in \Psi | \ \exists a > 0, \text{ such that } \forall u \in V, \ \varphi(u) = au\}$.

As a benchmark, we check whether these well-known invariance axioms are compatible with SP, AN and IIA. Previous studies have revealed that
(1) Neither INV-Φ^{IT} nor INV-Φ^{PAT} is compatible with SP, AN and IIA;
(2) While INV-Φ^{ST} is incompatible with SP, AN and IIA if $V = \mathbb{R}$, it is compatible if $V = \mathbb{R}_{++}$; and
(3) INV-Φ^{OT} is compatible with SP, AN and IIA.

What are the underlying reasons for such differences? The following figures give a graphical hint.

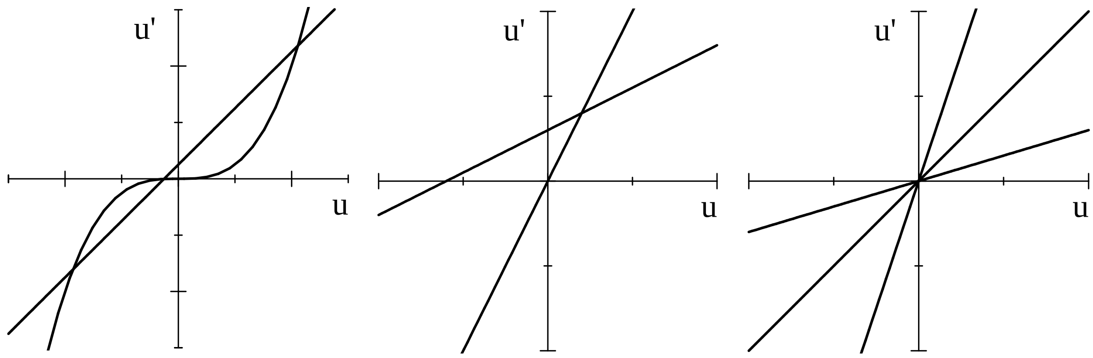

Figure 1. Φ^{IT}, Φ^{PAT} and Φ^{ST} (if $V = \mathbb{R}$) have an intersection.

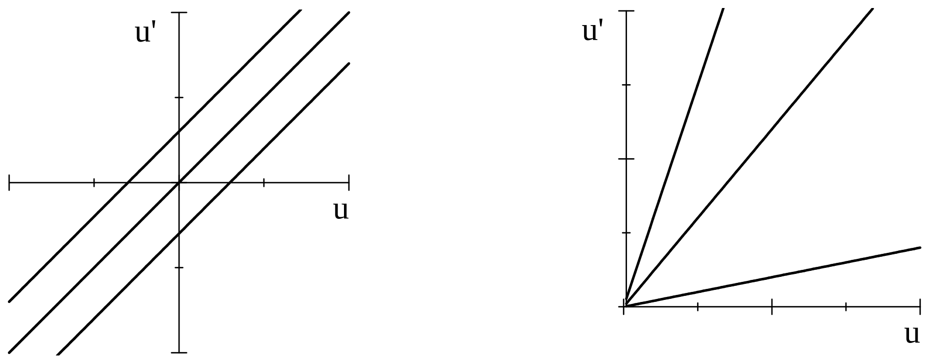

Figure 2. Φ^{OT} and Φ^{ST} (if $V = \mathbb{R}_{++}$) have no intersection.

We can observe a clear difference between Figures 1 and 2. While there are intersections in graphs of Figure 1, there is no intersection in the graphs of Figure 2. Actually, whether Φ has an intersection or not provides a necessary and sufficient condition for INV-Φ to be compatible with SP, AN and IIA. This property is defined as follows.

Non Crossing Property (NCP): *For each pair* $\varphi, \psi \in \Phi$, *if* $\varphi(u) \leq \psi(u)$, *for some* $u \in V$*, then* $\varphi(u) \leq \psi(u)$*, for each* $u \in V$*.*

Let $\leq$ be the binary relation defined over Φ, such that $\varphi \leq \psi$ if and only if $\varphi(u) \leq \psi(u)$, for each $u \in V$. For each pair $\varphi, \psi \in \Phi$, $\varphi \leq \psi$ means that *a change caused by* ψ is at least as large as *a changes caused by* φ. NCP requests that $\leq$ should be a *linear order*[11]. That is, with NCP it is possible to compare any pair of transformations in terms of *degrees of changes.*

In order to establish our results, we need to know the algebraic structure of Φ provided by NCP. It is described by the following lemma.

Lemma 1. *There is a function* $G : \Phi \to \mathbb{R}$ *such that for each pair* $\varphi, \psi \in \Phi$*,* $[\varphi \leq \psi, \text{ if and only if } G(\varphi) \leq G(\psi)]$ *and* $[G(\varphi \circ \psi) = G(\varphi) + G(\psi)]$ *if and only if* Φ *satisfies* NCP.

Proof. See appendix. ■

Lemma 1 says that Φ is order and group homomorphic to $\mathbb{R}$ if and only if Φ satisfies NCP. This algebraic structure makes it possible for the binary relation $\leq$ over Φ to be represented by a real valued function $G : \Phi \to \mathbb{R}$. According to this function G, for each $\varphi \in \Phi$, *a change caused by* φ is measured by $G(\varphi) \in \mathbb{R}$.

The way to measure *a change caused by* $\varphi \in \Phi$ is described as follows. First, let $\varphi_1 \in \Phi$ be such that for each $u \in V$, $\varphi_1(u) > u$. We regard *a change caused by* φ_1 as one unit. Second, for each $\varphi \in \Phi$ and each $m \in \mathbb{N}$, we say that *a change caused by* φ is as large as m units of φ_1 if and only if

$$\varphi = \underbrace{\varphi_1 \circ \cdots \circ \varphi_1}_{m\ times}.$$

Third, for each $\varphi \in \Phi$ and each pair $m, n \in \mathbb{Z}$, $n \neq 0$, we say that *a change caused*

[11] A binary relation $\leq$ over Φ is a *linear order* if $\leq$ satisfies *completeness* $[\forall \varphi, \psi \in \Phi,\ \varphi \leq \psi \text{ or } \psi \leq \varphi]$, *transitivity* $[\forall \varphi, \psi, \varsigma \in \Phi,\ \varphi \leq \psi \text{ and } \psi \leq \varsigma \Longrightarrow \varphi \leq \varsigma]$ and *antisymmetry* $[\forall \varphi, \psi \in \Phi,\ \varphi \leq \psi \text{ and } \psi \leq \varphi \Longrightarrow \varphi = \psi]$.

by φ is as large as $\frac{m}{n}$ units of φ_1 if and only if

$$\underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}} = \underbrace{\varphi_1 \circ \cdots \circ \varphi_1}_{m \text{ times}}.$$

Finally, for each $\varphi \in \Phi$, define $G(\varphi)$ as

$$G(\varphi) = \sup \left\{ \frac{m}{n} | \varphi^n \leq \varphi_1^m, \ n, m \in \mathbb{N} \right\}.$$

In the proof of Lemma 1, we check that this G is order and group homomorphic.

We need the following lemma which is known as "the Welfarism Theorem." Before introducing it, we define three axioms for SWOs.

Strong Pareto (SP): *For each pair* $u, u' \in V^n$, (1) *if for each* $i \in N$, $u_i \geq u'_i$, *then* $u \succsim_R u'$ *and* (2) *if for each* $i \in N$, $u_i \geq u'_i$ *and for some* $j \in N$, $u_j > u'_j$, *then* $u \succ_R u'$.

Anonymity (AN): *For each* $u \in V^n$ *and each permutation* π *of* N, $u \sim_R \pi(u)$.

Invariance with respect to Φ^N: *For each pair* $u, u' \in V^n$ *and each* $\varphi \in \Phi^N$, $u \succsim_R u'$ *if and only if* $\varphi(u) \succsim_R \varphi(u')$.

Theorem (Welfarism Theorem[12]): *If* a SWFL R *satisfies* SP *and* IIA, *then there is a* SWO $\succsim_R$ *such that for each* $U \in \mathcal{U}^N$ *and each pair* $x, y \in X$,

$$x \, R_U \, y \text{ if and only if } U(x) \succsim_R U(y).$$

Moreover, if R *satisfies* SP *and* AN *and* INV-Φ^N, *then* $\succsim_R$ *also satisfies* SP, AN *and* INV-Φ^N.

Now, we are ready to state part of our main result.

Theorem 1. *There is* a SWFL *which satisfies* SP, AN, IIA *and* INV-Φ *if and*

[12] See, for example, d'Aspremont and Gevers [10] and Hammond [18] .

only if Φ *satisfies* NCP.

Proof. Here we prove only the sufficiency part of Theorem 1 when Φ satisfies RC. The rest of the proof is relegated to the Appendix.

We use a function $G : \Phi \to \mathbb{R}$ defined as in Lemma 1. Let $u_0 \in V$ and $g : V \to \mathbb{R}$ be such that for each $u \in V$,

$$g(u) = G(\varphi), \text{ where } \varphi(u_0) = u.$$

Since Φ satisfies NCP and RC, g is well-defined. Let R be a transformed utilitarian rule associated with g. We show that R satisfies all the conditions in Theorem 1. Obviously, R satisfies AN and IIA.

Claim 1. *R satisfies SP.*

Proof. For each pair $u, u' \in V$, such that $u < u'$, let $\varphi, \psi \in \Phi$ be such that $\varphi(u_0) = u$, $\psi(u_0) = u'$. By the definitions of g and G,

$$g(u) = G(\varphi) < G(\psi) = g(u').$$

Thus, g is increasing, so that R satisfies SP.

Claim 2. *R satisfies* INV-Φ.

Proof. For each pair $x, y \in X$, each $i \in N$, and each $\varphi \in \Phi$, $x \, R_U \, y$, if and only if $\sum_{i \in N} g(U_i(x)) \geq \sum_{i \in N} g(U_i(y))$. This is equivalent to

$$\textstyle\sum_{i \in N} g(U_i(x)) + G(\varphi) \geq \sum_{i \in N} g(U_i(y)) + G(\varphi).$$

Let $\varphi^{U_i(x)} \in \Phi$ be such that $\varphi^{U_i(x)}(u_0) = U_i(x)$. Then,

$$\begin{aligned} g(\varphi \circ U_i(x)) &= G(\varphi \circ \varphi^{U_i(x)}) \\ &= G(\varphi) + G(\varphi^{U_i(x)}) \\ &= G(\varphi) + g(U_i(x)). \end{aligned}$$

Similarly,

$$g(\varphi \circ U_i(y)) = G(\varphi) + g(U_i(y)).$$

Hence, $x \, R_U \, y$ if and only if

$$\textstyle\sum_{j \neq i} g(U_j(x)) + g(\varphi \circ U_i(x)) \geq \sum_{j \neq i} g(U_j(y)) + g(\varphi \circ U_i(y)).$$

Therefore, $x \, R_U \, y$ if and only if $x \, R_{(\varphi \circ U_i, U_{-i})} \, y$.■

In order to interpret Theorem 1, we need to know what NCP requests in terms of *measurability* and *interpersonal comparability* of utility functions. Blackorby, Donaldson and Weymark [5] explain that *measurability* assumptions specify what transformations may be applied to an individual's utility function without altering the individually available information[13] and that *interpersonal comparability* assumptions specify how much of this information may be used interpersonally.

Let $\Phi \subseteq \Psi$ satisfy NCP and RC. For each pair $u, u' \in V$ and each $\varphi \in \Phi$, we say that *a change from u to u'* is *measured by φ* if $u' = \varphi(u)$. If Φ satisfies NCP and RC, then for each pair $i, j \in N$, each pair $u_i, u_i' \in V$, and each pair $u_j, u_j' \in V$, there is a unique pair $\varphi, \psi \in \Phi$ such that $\varphi(u_i) = u_i'$ and $\psi(u_j) = u_j'$. By NCP, either $\varphi \leq \psi$ or $\varphi \geq \psi$. Hence, we can compare a *change* from u_i to u_i' with a *change* from u_j to u_j' by comparing φ with ψ.

Remark 1. In order to use Φ as a measure of *degrees of changes*, any invariance transformation $\psi \in \Phi$ should not distort the information that a *change* from u to u' is measured by $\varphi \in \Phi$. That is, for each pair $u, u' \in V$, each $\psi \in \Phi$ and each $\varphi \in \Phi$, if $\varphi(u) = u'$, then $\varphi(\psi(u)) = \psi(u')$.

In fact, as long as Φ satisfies NCP, we do not need to worry about it. We say a group $(\Phi, \circ)$ is *commutative* if for each pair $\varphi, \psi \in \Phi$, $\varphi \circ \psi = \psi \circ \varphi$. If $(\Phi, \circ)$ is *commutative*, then since

$$\begin{aligned} \varphi(\psi(u)) &= \psi(\varphi(u)) \\ &= \psi(u'), \end{aligned}$$

a *change* from $\psi(u)$ to $\psi(u')$ is also measured by φ. Hence, we can state that if INV-Φ is imposed, then the information that a *change* from $U_i(x)$ to $U_i(y)$ is measured

[13] As long as $\mathcal{I}$ satisfies REP, we can directly quote their explanation.

by $\varphi \in \Phi$ is preserved.

Let us check that $(\Phi, \circ)$ is *commutative* if Φ satisfies NCP. Let $G : \Phi \rightarrow \mathbb{R}$ be a function defined as in Theorem 2. For each pair $\varphi, \psi \in \Phi$, since

$$\begin{aligned} G(\varphi \circ \psi) &= G(\varphi) + G(\psi) \\ &= G(\psi) + G(\varphi) = G(\psi \circ \varphi), \end{aligned}$$

we have both $\varphi \circ \psi \leq \psi \circ \varphi$ and $\varphi \circ \psi \geq \psi \circ \varphi$. Hence, $\varphi \circ \psi = \psi \circ \varphi$.■

The following examples illustrate several different measures through which we can compare *changes* of utility values interpersonally.

Example 1. Let $N = \{1, 2\}$, $X = \{x, y, z\}$ and $V = \mathbb{R}_{++}$. Let U_1 and U_2 be such that $U_1(x) = 1$, $U_1(y) = 4$, $U_2(x) = 9$, and $U_2(y) = 3$.
(1) When INV-Φ^{OT} is imposed, a *change* from $U_1(x) = 1$ to $U_1(y) = 4$ is measured by $\varphi_1(u) = u + 3$, and a *change* from $U_2(y) = 3$ to $U_2(x) = 9$ is measured by $\varphi_2(u) = u + 6$. Since $\varphi_1 < \varphi_2$, Φ^{OT} judges that a *change* from 3 to 9 is larger than a *change* from 1 to 4.
(2) When INV-Φ^{ST} is imposed, a *change* from $U_1(x) = 1$ to $U_1(y) = 4$ is measured by $\varphi_1(u) = 4u$ and a *change* from $U_2(y) = 3$ to $U_2(x) = 9$ is measured by $\varphi_2(u) = 3u$. Since $\varphi_1 > \varphi_2$, Φ^{ST} judges that a *change* from 1 to 4 is larger than a *change* from 3 to 9.

Example 2. Let $g \in \mathcal{G}$. For each $c \in \mathbb{R}$, let $\varphi_c : V \rightarrow V$ be such that for each $u \in V$, $\varphi_c(u) = g^{-1}(g(u) + c)$. We can easily check that (1) for each $c \in \mathbb{R}$, $\varphi_c \in \Psi$; and (2) for each pair $c, c' \in \mathbb{R}$, $c \leq c'$ if and only if $\varphi_c \leq \varphi_{c'}$. Hence, the set $\Phi^g \equiv \{\varphi_c \in \Psi : c \in \mathbb{R}\}$ satisfies NCP and RC[14].

When INV-Φ^g is imposed, the ethical observer can use Φ^g as a measure to compare *changes* of utility values interpersonally. For each pair $u, u' \in V$, we have

$$\varphi_{g(u')-g(u)}(u) = g^{-1}(g(u) + g(u') - g(u)) = u'.$$

[14]See, for example, Aczél [1] and Zdun [33].

Hence, a *change* from u to u' is measured by $\varphi_{g(u')-g(u)}$.

The next question is to consider what kind of SWFLs satisfy SP, AN IIA and INV-Φ (or INV-$\prod_{i \in N} \Phi$). The following theorem is quite important to capture the class of such SWFLs.

Theorem 2. *Suppose* Φ *satisfies* RC. *Then, there is at most one* SWFL *which satisfies* SP*,* AN*,* IIA *and* INV-Φ.
Proof. See Appendix.■

The following example shows that the previous statement does not necessarily hold when RC is not imposed.

Example 3. Let $\Phi = \{\varphi \in \Psi | \ \exists b \in \mathbb{Z} \text{ such that } \forall u \in V, \ \varphi(u) = u + b\}$. Let $g : V \to \mathbb{R}$ be such that for each $u \in V$,

$$g(u) = [u] + (u - [u])^a,$$

where $[u] \equiv \max\{u' \in \mathbb{Z} | u' \leq u\}$ and $a > 0$. Then, the transformed utilitarian rule associated with g, satisfies SP, AN, IIA and INV-Φ. This is because for each $\varphi(u) = u + b \in \Phi$, $b \in \mathbb{Z}$, and each $u \in V$,

$$\begin{aligned} g(\varphi(u)) &= [u+b] + (u+b-[u+b])^a \\ &= [u] + b + (u - [u])^a \\ &= g(u) + b. \end{aligned}$$

If $a \neq 1$, this transformed utilitarian rule is not the utilitarian rule.■

Theorem 2 says that if a *rich* Φ satisfies NCP, then there is a unique SWFL satisfying SP, AN, IIA and INV-Φ. Recall that in the proof of Theorem 1, for each $\Phi \subseteq \Psi$, satisfying NCP and RC, we show the existence of a transformed utilitarian rule satisfying SP, AN, IIA and INV-Φ. Hence, Theorems 1 and 2 together imply that if Φ satisfies NCP and RC, then SP, AN, IIA and INV-Φ provides a characterization of a particular transformed utilitarian rule.

Conversely, for each $g \in \mathcal{G}$, we can characterize the transformed utilitarian rule associated with g, based on some $\Phi \subseteq \Psi$. Let

$$\Phi^g \equiv \left\{\varphi \in \Psi : \exists c \in \mathbb{R} \text{ such that } \forall u \in V,\, \varphi(u) = g^{-1}(g(u) + c)\right\}$$

be the subset of increasing transformations defined as in Example 2. Then, since for each $\varphi \in \Phi^g$, there is $c \in \mathbb{R}$ such that for each $u \in V$,

$$g(\varphi(u)) = g(g^{-1}(g(u) + c)) = g(u) + c,$$

the transformed utilitarian rule associated with g satisfies INV-Φ^g. Hence, by Theorem 2, it is characterized by SP, AN, IIA and INV-Φ^g.

These facts are summarized by the following proposition.

Proposition 2.
(1) *For each* $g \in \mathcal{G}$, *the transformed utilitarian rule associated with* g *is characterized by* SP, AN, IIA *and* INV-Φ^g.
(2) *Suppose that* R *satisfies* SP, AN *and* IIA. *Then,* R *is a transformed utilitarian rule if and only if there is a rich* $\Phi \subseteq \Psi$ *such that* R *satisfies* INV-Φ.
(3) *Suppose that* Φ *satisfies* RC. *There is* a SWFL *which satisfies* SP, AN, IIA *and* INV-Φ, *if and only if there is* $g \in \mathcal{G}$ *such that* $\Phi = \Phi^g$.

We end this section by checking whether well-known invariance axioms are compatible with SP, AN and IIA or not. The following results are obtained as corollaries of Proposition 2.

Corollary 1.
(1) *No* SWFL *satisfies* SP, AN, IIA *and* INV-Φ^{IT} (resp. INV-Φ^{PAT}) (Arrow [2], Sen [27].)
(2) *Only the utilitarian rule satisfies* SP, AN, IIA *and* INV-Φ^{OT} (d'Aspremont and Gevers [10].)
(3) *If* $V = \mathbb{R}$, *no* SWFL *satisfies* SP, AN, IIA *and* INV-Φ^{ST}. *If* $V = \mathbb{R}_{++}$, *only*

the Nash rule satisfies SP, AN, IIA *and* INV-Φ^{ST} (Moulin [20], Tsui and Weymark [32].)

4.2 General Case

While Theorem 1 considers a large class of informational structures, it does not cover an important informational structure called cardinal and unit comparability (CUC), which is represented by

$$\Phi^{CUC} \equiv \left\{\varphi \in \Psi^N | \exists a > 0, \exists b \in \mathbb{R}^n, \text{ such that } \forall i \in N, \ \forall u \in V, \ \varphi_i(u) = au + b_i, \right\}.$$

It is well known that CUC makes interpersonal comparisons of utility *differences* possible and that INV-Φ^{CUC} provides a characterization of the utilitarian rule. Here we study the class of informational structures satisfying SYM, REP and SEP, which contains CUC.

In order to establish our main result, it is useful to identify the necessary information for the utilitarian rule.

Proposition 3. *The utilitarian rule satisfies* INV-Φ^N *if and only if* $\Phi^N \subseteq \Phi^{CUC}$.
Proof. See Appendix.■

With INV-Φ^N such that $\Phi^N \subseteq \Phi^{CUC}$, utility functions are also cardinal and unit comparable. Hence, Proposition 3 reveals the necessity of cardinal and unit comparability for the utilitarian rule. Likewise, for each transformed utilitarian rule, we can specify what kind of information is necessary to use it. For each $g \in \mathcal{G}$, let

$$\begin{aligned}\Phi^{CUC-g} &\equiv \left\{\varphi \in \Psi^N | \exists a > 0, \exists b \in \mathbb{R}^n, \text{ such that } \forall i \in N, \ \forall u \in V, \ \varphi_i(u) = g^{-1}(ag(u) + b_i)\right\} \\ &= \left\{\varphi \in \Psi^N | \exists a > 0, \exists b \in \mathbb{R}^n, \text{ such that } \forall i \in N, \ \forall u \in V, \ g(\varphi_i(u)) = ag(u) + b_i\right\}.\end{aligned}$$

We call the informational structure represented by Φ^{CUC-g} cardinal and unit comparability of transformed utilities by g (CUC-g). Like CUC, CUC-g makes transformed utility functions $(g \circ U_1, \ldots, g \circ U_n)$ cardinal and unit comparable. Hence, *differences* of transformed utilities by g are interpersonally comparable. Actually, for each $\varphi \in \Phi^{CUC-g}$, each pair $i, j \in N$ and each quadruplet $x, y, w, z \in A$, we can easily check that

$$g(U_i(x)) - g(U_i(y)) \geq g(U_j(w)) - g(U_j(z))$$

if and only if

$$g(\varphi_i(U_i(x))) - g(\varphi_i(U_i(y))) \geq g(\varphi_j(U_j(w))) - g(\varphi_j(U_j(z))).$$

We illustrate one informational structure which is represented by Φ^{CUC-g}.

Example 4. Let $V = \mathbb{R}_{++}$. Consider the informational structure called ratio scale measurability (RSM), which is represented by

$$\Phi^{RSM} \equiv \prod_{i \in N} \Phi^{ST} = \left\{\varphi \in \Psi^N | \exists b \in \mathbb{R}^n_{++}, \text{ such that } \forall i \in N,\ \forall u \in V,\ \varphi_i(u) = b_i u\right\}.$$

RSM has been regarded as an informational structure which permits interpersonal comparisons of *ratios of utilities*. In fact, there is a larger class of transformations which preserve such information. Let

$$\begin{aligned}\Phi^{CUC-\log} &\equiv \left\{\varphi \in \Psi^N | \exists a > 0, \exists b \in \mathbb{R}^n, \text{ such that } \forall i \in N,\ \forall u \in V,\ \varphi_i(u) = \log^{-1}(a \log u + b_i)\right\} \\ &= \left\{\varphi \in \Psi^N | \exists a > 0, \exists b \in \mathbb{R}^n_{++}, \text{ such that } \forall i \in N,\ \forall u \in V,\ \varphi_i(u) = b_i u^a\right\}.\end{aligned}$$

While $\Phi^{RSM} \subsetneq \Phi^{CUC-\log}$ and CUC-log still makes it possible to compare ratios of utilities interpersonally.■

Remark 2. For each $g \in \mathcal{G}$, we have proposed two ways to measure changes of utility values with g. One way is to use Φ^g defined as in Example 2. According to Φ^g, for each pair $u, u' \in V$, a *change* from u to u' is measured as $\varphi_{g(u')-g(u)} \in \Phi^g$. The other way proposed here is to take differences of transformed utility values by g. Then, for each pair $u, u' \in V$, a *change* from u to u' is measured by $g(u') - g(u) \in \mathbb{R}$.

Indeed, these two ways measure changes of utility values in the same way. This is because for each quadruplet $u_i, u_i', u_j, u_j' \in V$, $g(u_i') - g(u_i) \leq g(u_j') - g(u_j)$ if and only if $\varphi_{g(u_i')-g(u_i)} \leq \varphi_{g(u_j')-g(u_j)}$.■

The following theorem shows that in order to use the transformed utilitarian rule associated with g, transformed utility functions $(g \circ U_1, \ldots, g \circ U_n)$ need to be cardinal and unit comparable.

Proposition 4. *Let $g \in \mathcal{G}$. The transformed utilitarian rule associated g satisfies* INV-Φ^N *if and only if* $\Phi^N \subseteq \Phi^{CUC-g}$.
Proof. The proof is similar to that of Proposition 3.■

Now, we are ready to state our main result, which is a characterization of informational structures with which SP, AN and IIA are consistent.

Theorem 3. *Let $\mathcal{I}$ satisfy* SYM, REP *and* SEP *and* Φ^N *represent $\mathcal{I}$. There is* a SWFL *which satisfies* SP, AN, IIA *and* INV-$\mathcal{I}$ *if and only if there is $g \in \mathcal{G}$ such that* $\Phi^N \subseteq \Phi^{CUC-g}$.
Proof. By Proposition 1, there is a rich $\Phi \subseteq \Psi$ such that $\prod_{i \in N} \Phi \subseteq \Phi^N$. Since INV-$\Phi^N$ implies INV-Φ, by Theorems 1, 2 and Proposition 2, a SWFL satisfying all the condition in Theorem 3 must be a transformed utilitarian rule. Hence, by Proposition 4, there is $g \in \mathcal{G}$ such that $\Phi^N \subseteq \Phi^{CUC-g}$. ■

Theorem 3 specifies the information necessary for the existence of Arrovian SWFLs. It says that when SEP is imposed as an informational assumption, an invariance axiom is compatible with SP, AN and IIA if and only if it permits cardinal and unit comparability of transformed utility functions by a certain transformation $g \in \mathcal{G}$. That is, the ethical observer needs a transformation $g : V \to \mathbb{R}$ through which he can treat transformed utility functions $(g \circ U_1, \ldots, g \circ U_n)$ as cardinal and unit comparable. Recall that the prohibition of interpersonal *level* comparison implies SEP. Hence, Theorem 3 tells us that as long as interpersonal *level* comparisons are prohibited, transformed utility functions by a certain transformation need to be cardinal and unit comparable. Therefore, we conclude that either ordinal and level comparability or cardinal and unit comparability of transformed utility functions by a certain transformation is necessary for Arrovian aggregation.

We end this section by providing a characterization of transformed utilitarian rules, which is a by-production of Theorem 3. This says that if interpersonal *level* comparison is prohibited, then only transformed utilitarian rules are available Arrovian SWFLs.

Proposition 5.
(1) *For each* $g \in \mathcal{G}$, *the transformed utilitarian rule associated with* g *is characterized by* SP, AN, IIA *and* INV-Φ^{CUC-g}.
(2) *Suppose that* R *satisfies* SP, AN *and* IIA. R *is a transformed utilitarian rule if and only if there is* $\mathcal{I}$, *satisfying* SYM, REP *and* SEP, *such that* R *satisfies* INV-$\mathcal{I}$.

5 Conclusion

While studies of SWFLs have a long history, originated by Sen [27], we have proposed a new approach to this field. Many of major previous results were characterizations of certain SWFLs. An informational structure has been treated as a fixed assumption to obtain characterizations of SWFLs. However, utility theory has not reached a consensus on actually available information for utility functions, so that an invariance axiom, which is imposed to obtain characterizations of SWFLs, may not be reasonable. Contrary to such previous studies, our approach is independent of the dispute about observable utility information. This is because we do not treat an invariance axiom as a fixed assumption. We just aim to capture the feature of informational structures necessary to obtain some possibility results.

Our main result is a characterization of the class of infornational structures with which *Strong Pareto*, *Anonymity* and *Independence of Irrelevant Alternatives* are consistent. It says that such an invariance axiom requests transformed utility functions by a certain transformation to be cardinal and unit comparable, if interpersonal *level* comparisons are prohibited. Therefore, we conclude that either ordinal and level comparability or cardinal and unit comparability of transformed utility functions by a certain transformation is necessary for Arrovian aggregation.

We end this paper by proposing some open problems. In this paper, we have considered only the compatibility with *Strong Pareto*, *Anonymity* and *Independence of Irrelevant Alternatives*, so it is also necessary to consider what kind of invariance axioms are compatible with some other axioms. Especially, it is worth studying how much informational requirements can be dropped if *Strong Pareto* and *Anonymity* are weakened to *Weak Pareto* and *Non Dictatorship*, respectively. We also need to consider the variable population case.

Invariance axioms have been considered in other social choice problems, such as the bargaining problem (Nash [21]; Shapley [31]), cooperative games (Shapley [31]), and the problem of ranking opportunity sets (Pattanaik and Xu [22]). It may be interesting to study what kind of invariance axioms are compatible with some other natural axioms in such problems.

6 Appendix: Proofs

6.1 Proof of Proposition 1

We introduce the following notation. For each $i \in N$ and each $\varphi \in \Psi$, define $\varphi^i \in \Psi^N$ by for $i \in N$, $\varphi_i = \varphi$ and for each $j \neq i$, $\varphi_j = id_v$.

Necessity: It suffices to show that there is a rich $\Phi \subseteq \Psi$, such that for each $i \in N$ and each $\varphi \in \Phi$, $\varphi^i \in \Phi^N$.

For each $i \in N$ and each pair $u, u' \in V$, let $U, U' \in \mathcal{U}^N$ be such that for each $x \in A$, $U_i(x) = u$ and $U'_i(x) = u'$; and for each $j \neq i$, $v_j = v'_j$. By SEP, $U \, \mathcal{I} \, U'$, so that there is $\varphi \in \Phi^N$ such that $\varphi \circ U = U'$.

Since for each $j \neq i$, $U_j = U'_j = \varphi_j \circ v_j$, we have for each $j \neq i$, $\varphi_j = id_v$. Hence, there is $\varphi \in \Psi$ such that $\varphi^i \in \Phi^N$ and

$$\varphi(u) = \varphi(U_i(x)) = U'_i(x) = u'.$$

Sufficiency: Suppose that there is a rich $\Phi \subseteq \Psi$ such that $\prod_{i \in N} \Phi \subseteq \Phi^N$. Let $U, U' \in \mathcal{U}^N$ be such that there is $S \subseteq N$, such that for each $i \in S$ and each $x \in A$, $U_i(x) = U'_i(x)$; and for each $i \in N \backslash S$ and each pair $x, y \in A$, $U_i(x) = U_i(y)$ and $U'_i(x) = U'_i(y)$. For each $i \in S$, let $\varphi_i = id_v \in \Phi$. Then, $\varphi_i \circ U_i = U_i = U'_i$. For each $i \in N \backslash S$, let $\varphi_i \in \Phi$ be such that $\varphi_i \left(U_i(x)\right) = U'_i(x)$. Then, since for each pair

$x, y \in A$, $U_i(x) = U_i(y)$ and $U'_i(x) = U'_i(y)$, for each $y \in A$,

$$\begin{aligned} \varphi_i \left(U_i(y) \right) &= \varphi_i \left(U_i(x) \right) \\ &= U'_i(x) = U'_i(y). \end{aligned}$$

Therefore, for $\varphi = (\varphi_1, \ldots, \varphi_n) \in \prod_{i \in N} \Phi \subseteq \Phi^N$, we have $\varphi \circ U = U'$, so that $U \mathcal{I} U'$.■

6.2 Proof of Lemma 1

We introduce some notation.

For each $n \in \mathbb{N}$ and each $\varphi \in \Phi$, define φ^n by

$$\varphi^n \equiv \underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}}.$$

Given Φ satisfying NCP, define $\Phi^+ \subseteq \Phi$ by

$$\Phi^+ \equiv \{\varphi \in \Phi | \text{ for each } u \in V, \ \varphi(u) \geq u\}.$$

Step 1. *For each $\varphi \in \Phi$, φ is continuous.*

Proof. Suppose that there is $\varphi \in \Phi$ that is not continuous. Without loss of generality, let

$$\lim_{u \to u^* - 0} \varphi(u) < \varphi(u^*).$$

Since Φ is a subgroup, there is $\varphi^{-1} \in \Phi$ such that $\varphi^{-1} \circ \varphi = id_v$. Let $\widetilde{u} \in \left(\lim_{u \to u^* - 0} \varphi(u), \varphi(u^*) \right)$. Then, since for each $u' < u^*$, $\widetilde{u} > \varphi(u')$, $\varphi^{-1}(\widetilde{u}) > \varphi^{-1} \circ \varphi(u') = u'$. Hence, $\varphi^{-1}(\widetilde{u}) \geq u^*$. However, since $\widetilde{u} < \varphi(u^*)$, $\varphi^{-1}(\widetilde{u}) < \varphi^{-1}(\varphi(u^*)) = u^*$, which is a contradiction.■

Step 2. *Suppose that Φ satisfies* NCP. *Then, for each $\varphi \in \Phi^+ \backslash \{id_V\}$ and each $u \in V$,*

$$\lim_{n \to \infty} \varphi^n(u) = \infty.$$

Proof. Suppose that there are $\varphi \in \Phi^+ \backslash \{id_V\}$ and $u \in V$, such that $\lim_{n\to\infty} \varphi^n(u) < \infty$. Since $\{\varphi^n(u)\}_{n\in\mathbb{N}}$ is a monotone increasing sequence, there is u^* such that

$$\lim_{n\to\infty} \varphi^n(u) = u^*.$$

By Step 1, φ is continuous, so that

$$\varphi(u^*) = \lim_{n\to\infty} \varphi(\varphi^n(u)) = \lim_{n\to\infty} \varphi^{n+1}(u) = u^*.$$

Hence, by NCP, $\varphi = id_V$, which contradicts $\varphi \in \Phi^+ \backslash \{id_V\}$.■

Step 3. *Suppose that* Φ *satisfies* NCP. *Then, for each pair* $\varphi, \varphi' \in \Phi^+ \backslash \{id_V\}$, *there is* $n \in \mathbb{N}$ *such that*

$$\varphi^n \geq \varphi'.$$

Proof. Let $u \in V$. By Step 2, there is $n \in \mathbb{N}$ such that

$$\varphi^n(u) \geq \varphi'(u).$$

Then, by NCP, for each $u' \in V$,

$$\varphi^n(u') \geq \varphi'(u').■$$

Let $\varphi_1 \in \Phi^+ \backslash \{id_V\}$ and for each $\varphi \in \Phi^+$, define $G(\varphi) \in \mathbb{R}$ by

$$G(\varphi) \equiv \sup \left\{ \frac{m}{n} | \varphi^n \leq \varphi_1^m, \ n, m \in \mathbb{N} \right\}.$$

By, Step 3, we can easily check that for each $\varphi \in \Phi^+ \backslash \{id_V\}$, $G(\varphi) \in \mathbb{R}_{++}$.

Step 4. *For each pair* $\varphi, \psi \in \Phi^+$, *if* $\varphi \leq \psi$, *then* $G(\varphi) \leq G(\psi)$.

Proof. For each pair $n, m \in \mathbb{N}$, if $\psi^n \leq \varphi_1^m$, then $\varphi^n \leq \psi^n \leq \varphi_1^m$. Hence,

$$\begin{aligned} G(\varphi) &= \sup\left\{\frac{m}{n} | \varphi^n \leq \varphi_1^m,\ n, m \in \mathbb{N}\right\} \\ &\leq \sup\left\{\frac{m}{n} | \psi^n \leq \varphi_1^m,\ n, m \in \mathbb{N}\right\} = G(\psi). \blacksquare \end{aligned}$$

Step 5. *For each pair* $\varphi, \psi \in \Phi^+$, $G(\varphi \circ \psi) = G(\psi \circ \varphi)$.

Proof. We distinguish two cases.

Case 1: $\min\left\{G(\varphi) | \varphi \in \Phi_i^+ \setminus \{id_V\}\right\}$ exists.

Let $\varphi_1 \in \arg\min\{G(\varphi) | \varphi \in \Phi^+ \setminus \{id_V\}\}$. If there are $\varphi \in \Phi^+ \setminus \{id_V\}$ and $n \in \mathbb{N}$ such that $\varphi_1^n < \varphi < \varphi_1^{n+1}$, then since

$$id_v = \varphi_1^{-n} \circ \varphi_1^n < \varphi_1^{-n} \circ \varphi < \varphi_1^{-n} \circ \varphi_1^{n+1} = \varphi_1,$$

we have $\varphi_1^{-n} \circ \varphi \in \Phi^+ \setminus \{id_V\}$ and $\varphi_1^{-n} \circ \varphi < \varphi_1$, which is a contradiction. Hence, for each pair $\varphi, \psi \in \Phi^+ \setminus \{id_V\}$, there are $n, n' \in \mathbb{N}$ such that

$$\varphi = \varphi_1^n, \text{ and } \psi = \varphi_1^{n'}.$$

Therefore,

$$G(\varphi \circ \psi) = G(\varphi_1^{n+n'}) = G(\psi \circ \varphi).$$

Case 2: $\min\{G(\varphi) | \varphi \in \Phi^+ \setminus \{id_V\}\}$ does not exist.

Suppose that $\inf\{G(\varphi) | \varphi \in \Phi^+ \setminus \{id_V\}\} > 0$. Then, there are $\varphi, \psi \in \Phi^+ \setminus \{id_V\}$ such that $\varphi < \psi$ and

$$G(\varphi) \leq G(\psi) < 2 \inf\left\{G(\varphi) | \varphi \in \Phi^+ \setminus \{id_V\}\right\}.$$

Since $\varphi^{-1} \circ \psi > id_V$, there is $\xi \in \Phi^+ \setminus \{id_V\}$ such that $\xi < min\{\varphi, \varphi^{-1} \circ \psi\}$. Then, by Step 4,

$$
\begin{aligned}
G(\psi) &= G(\varphi \circ (\varphi^{-1} \circ \psi)) \\
&\geq G(\xi \circ \xi) \\
&= 2G(\xi) \\
&> 2 \inf \left\{ G(\varphi) | \varphi \in \Phi^{+} \backslash \{id_V\} \right\},
\end{aligned}
$$

which is a contradiction.

Hence, for each $\epsilon > 0$, there is $\varphi_\epsilon \in \Phi^{+} \backslash \{id_V\}$ such that $G(\varphi_\epsilon) < \epsilon$. For each pair $\varphi, \psi \in \Phi^{+} \backslash \{id_V\}$, by Step 3, there are $n, n' \in \mathbb{N}$ such that

$$\varphi_\epsilon^{n} \leq \varphi < \varphi_\epsilon^{n+1} \text{ and } \varphi_\epsilon^{n'} \leq \psi < \varphi_\epsilon^{n'+1}.$$

Then, since

$$\varphi_\epsilon^{n+n'} \leq \varphi \circ \psi < \varphi_\epsilon^{n+n'+2} \text{ and } \varphi_\epsilon^{n+n'} \leq \psi \circ \varphi < \varphi_\epsilon^{n+n'+2},$$

we have

$$(n+n')G(\varphi_\epsilon) \leq G(\varphi \circ \psi) \leq (n+n'+2)G(\varphi_\epsilon),$$

$$(n+n')G(\varphi_\epsilon) \leq G(\psi \circ \varphi) \leq (n+n'+2)G(\varphi_\epsilon),$$

from which we obtain

$$|G(\varphi \circ \psi) - G(\psi \circ \varphi)| \leq 2G(\varphi_\epsilon) < 2\epsilon.$$

Therefore, $G(\varphi \circ \psi) = G(\psi \circ \varphi)$.■

Step 6. *For each pair* $\varphi, \psi \in \Phi^{+}$, $G(\varphi \circ \psi) = G(\varphi) + G(\psi)$.
Proof. It is sufficient to show that for each pair $\varphi, \psi \in \Phi^{+}$ and each quadruplet $n, m, n', m' \in \mathbb{N}$,

$$G(\varphi) \leq \frac{m}{n} \text{ and } G(\psi) \leq \frac{m'}{n'} \text{ together imply } G(\varphi \circ \psi) \leq \frac{m}{n} + \frac{m'}{n'}, \text{ and}$$

$$G(\varphi) \geq \frac{m}{n} \text{ and } G(\psi) \geq \frac{m'}{n'} \text{ together imply } G(\varphi \circ \psi) \geq \frac{m}{n} + \frac{m'}{n'}.$$

If $G(\varphi) \leq \frac{m}{n}$ and $G(\psi) \leq \frac{m'}{n'}$, then by the definition of G, we have $\varphi^n \leq \varphi_1^m$ and $\psi^{n'} \leq \varphi_1^{m'}$. Then, since $\varphi^{nn'} \leq \varphi_1^{n'm}$ and $\psi^{nn'} \leq \varphi_1^{nm'}$,

$$\varphi^{nn'} \circ \psi^{nn'} \leq \varphi_1^{n'm+nm'}.$$

Hence, by Step 5,

$$\begin{aligned} nn'G(\varphi \circ \psi) &= G((\varphi \circ \psi)^{nn'}) \\ &= G(\varphi^{nn'} \circ \psi^{nn'}) \\ &\leq G(\varphi_1^{n'm+nm'}) \\ &= n'm + nm', \end{aligned}$$

from which we obtain

$$G(\varphi \circ \psi) \leq \frac{m}{n} + \frac{m'}{n'}.$$

We can similarly show that $G(\varphi) \geq \frac{m}{n}$ and $G(\psi) \geq \frac{m'}{n'}$ together imply $G(\varphi \circ \psi) \geq \frac{m}{n} + \frac{m'}{n'}$.■

For each $\varphi \in \Phi \backslash \Phi^+$, define $G(\varphi) \in \mathbb{R}$ by

$$G(\varphi) = -G(\varphi^{-1}),$$

where $\varphi^{-1} \circ \varphi = id_V$. For each $\varphi \in \Phi \backslash \Phi^+$, since $\varphi^{-1} \in \Phi^+$, we have $G(\varphi) \leq 0$.

Step 7. *For each pair* $\varphi, \psi \in \Phi$, $G(\varphi \circ \psi) = G(\varphi) + G(\psi)$.
Proof. If $\varphi \circ \psi \in \Phi^+$ and $\varphi \in \Phi \backslash \Phi^+$, and then by Step 6,

$$\begin{aligned} G(\psi) &= G(\varphi^{-1} \circ \varphi \circ \psi) = G(\varphi^{-1}) + G(\varphi \circ \psi) \\ &= -G(\varphi) + G(\varphi \circ \psi), \end{aligned}$$

from which we obtain $G(\varphi \circ \psi) = G(\varphi) + G(\psi)$. We can similarly show that if $\varphi \circ \psi \in \Phi \backslash \Phi^+$, then $G(\varphi \circ \psi) = G(\varphi) + G(\psi)$.■

Step 8. *For each pair $\varphi, \psi \in \Phi_i$, $\varphi \leq \psi$ if and only if $G(\varphi) \leq G(\psi)$.*

Proof. Let be $\varphi, \psi \in \Phi$ such that $\varphi < \psi$. Then, since $\varphi^{-1} \circ \psi \in \Phi^{+} \backslash \{id_V\}$, we have $G(\varphi^{-1} \circ \psi) > 0$. Then, by Step 7,

$$\begin{aligned} G(\psi) &= G(\varphi \circ \varphi^{-1} \circ \psi) = G(\varphi) + G(\varphi^{-1} \circ \psi) \\ &> G(\varphi). \blacksquare \end{aligned}$$

Step 9. *Suppose there is a mapping $G : \Phi \to \mathbb{R}$ such that for each pair $\varphi, \psi \in \Phi$, $[\varphi \leq \psi, \text{ if and only if } G(\varphi) \leq G(\psi)]$ and $[G(\varphi \circ \psi) = G(\varphi) + G(\psi)]$, then Φ satisfies* NCP.

Proof. For each pair $\varphi, \psi \in \Phi$, since either $G(\varphi) \leq G(\psi)$ or $G(\varphi) \geq G(\psi)$, either $\varphi \leq \psi$ or $\varphi \geq \psi$ also holds by the definition of G. Hence, Φ satisfies NCP.

6.3 Proof of Theorem 1

Necessity: Suppose that Φ does not satisfy NCP and that there is a SWFL R which satisfies SP, AN, IIA and INV-Φ. Then, there are $\varphi_1, \varphi_2 \in \Phi$, and $u_1, u_2 \in V$ such that

$$[\varphi_1(u_1) > \varphi_2(u_1), \text{ and } \varphi_1(u_2) \leq \varphi_2(u_2)] \text{ or } [\varphi_1(u_1) < \varphi_2(u_1), \text{ and } \varphi_1(u_2) \geq \varphi_2(u_2)].$$

Without loss of generality, suppose that $\varphi_1(u_1) > \varphi_2(u_1)$ and $\varphi_1(u_2) \leq \varphi_2(u_2)$. Let $U \in \mathcal{U}^N$ be such that

$$U_1(x) = U_2(y) = u_1, \; U_1(y) = U_2(x) = u_2 \text{ and } U_i(x) = U_i(y) = u, \text{ for each } i \in N \backslash \{1, 2\}.$$

By the Welfarism Theorem, there is $\succsim_R \subset V^N \times V^N$ such that for each $U \in \mathcal{U}^N$ and each pair $x, y \in X$, $x \, R_U \, y$ if and only if $U(x) \succsim_R U(y)$. By AN of $\succsim_R$,

$$(u_1, u_2, u, \cdots, u) \sim_R (u_2, u_1, u, \cdots, u),$$

so that $x\, I_U\, y$. By INV-Φ, for each pair $\varphi_1, \varphi_2 \in \Phi$, we have $x\, I_{(\varphi_1 \circ U_1, \varphi_2 \circ U_2, U_{N \setminus \{1,2\}})}\, y$. Hence,

$$(\varphi_1(u_1), \varphi_2(u_2), u, \cdots, u) \sim_R (\varphi_1(u_2), \varphi_2(u_1), u, \cdots, u).$$

By AN of $\succsim_R$,

$$(\varphi_1(u_2), \varphi_2(u_1), u, \cdots, u) \sim_R (\varphi_2(u_1), \varphi_1(u_2), u, \cdots, u).$$

Therefore,

$$(\varphi_1(u_1), \varphi_2(u_2), u, \cdots, u) \sim_R (\varphi_2(u_1), \varphi_1(u_2), u, \cdots, u).$$

However, since $\varphi_1(u_1) > \varphi_2(u_1)$ and $\varphi_2(u_2) \geq \varphi_2(u_1)$, by SP of $\succsim_R$, we have

$$(\varphi_1(u_1), \varphi_2(u_2), u, \cdots, u) \succ_R (\varphi_2(u_1), \varphi_1(u_2), u, \cdots, u),$$

which is a contradiction.■

Sufficiency: Here we consider the case where Φ is not rich. We distinguish two cases in the same way with the proof of Step 5 of Lemma 1.

Case 1: $\min \{G(\varphi) | \varphi \in \Phi^+ \setminus \{id_V\}\}$ exists.
Let $\varphi_1 = \min \{G(\varphi) | \varphi \in \Phi^+ \setminus \{id_V\}\}$ and $u_0 \in V$. Define $g : V \to \mathbb{R}$ in the following way. First, for each $n \in \mathbb{Z}$, let

$$g(\varphi_1^n(u_0)) \equiv n.$$

Then, for $u_0, \varphi_1(u_0) \in V$, we have $g(u_0) = 0$ and $g(\varphi_1(u_0)) = 1$. Next, for each $u \in [u_0, \varphi_1(u_0)]$, let

$$g(u) \equiv \frac{u - u_0}{\varphi_1(u_0) - u_0}.$$

Obviously, g is continuous and increasing in $[u_0, \varphi_1(u_0)]$. Finally, for each $u \in V$, let $n \in \mathbb{Z}$ and $u' \in [u_0, \varphi_1(u_0)]$ be such that $u = \varphi_1^n(u')$, and

$$g(u) \equiv n + g(u').$$

We can easily check that this function $g : V \to \mathbb{R}$ is well-defined.

Let R be a transformed utilitarian rule associated with g. We show that R satisfies all the conditions in Theorem 1. It is obvious that R satisfies AN and IIA.

First, let us show that R satisfies SP. To do so, it suffices to show that g is increasing. By the definition of g, g is increasing in $[u_0, \varphi_1(u_0)]$. Then, since for each pair $u, u' \in [\varphi_1(v_0), \varphi_1^2(v_0)]$, $u < u'$,

$$\begin{aligned} g(u) &= 1 + g(\varphi^{-1}(u)) \\ &< 1 + g(\varphi^{-1}(u')) \\ &= g(u'). \end{aligned}$$

g is also increasing in $[\varphi_1(v_0), \varphi_1^2(v_0)]$. Taking similar steps, we can show that g is increasing everywhere.

Next, we show that R satisfies INV-**Φ**. For each pair $x, y \in X$, each $i \in N$, and each $\varphi = \varphi_1^n \in \Phi$, $x\, R_U\, y$, if and only if $\sum_{i \in N} g(U_i(x)) \geq \sum_{i \in N} g(U_i(y))$. This is equivalent to

$$\textstyle\sum_{i \in N} g(U_i(x)) + n \geq \sum_{i \in N} g(U_i(y)) + n.$$

By the definitions of φ,

$$\begin{aligned} g(\varphi \circ U_i(x)) &= g(\varphi_1^n \circ U_i(x)) \\ &= n + g(U_i(x)). \end{aligned}$$

Similarly,

$$g(\varphi \circ U_i(y)) = n + g(U_i(y)).$$

Hence, $x\, R_U\, y$ if and only if

$$\textstyle\sum_{j \neq i} g(U_j(x)) + g(\varphi \circ U_i(x)) \geq \sum_{j \neq i} g(U_j(y)) + g(\varphi \circ U_i(y)).$$

Therefore, $x\, R_U\, y$ if and only if $x\, R_{(\varphi \circ U_i, U_{-i})}\, y$.■

Case 2: $\min \{G(\varphi) | \varphi \in \Phi^+ \setminus \{id_V\}\}$ does not exist.
Let $G : \Phi \to \mathbb{R}$ be a mapping defined as in Lemma 1. Let $u_0 \in V$. For each $u \in V$,

let

$$g(u) \equiv \sup \{G(\varphi)|\varphi(u_0) \leq u\}.$$

Step 1. *g is non-decreasing.*

Proof. Let $u, u' \in V$ be such that $u \leq u'$. Since

$$\{G(\varphi)|\varphi(v_0) \leq u\} \subseteq \{G(\varphi)|\varphi(u_0) \leq u'\},$$

we have

$$\begin{aligned} g(u) &= \sup \{G(\varphi)|\varphi(u_0) \leq u\} \\ &\leq \sup \{G(\varphi)|\varphi(u_0) \leq u'\} \leq g(u'). \blacksquare \end{aligned}$$

Step 2. *g is continuous.*

Proof. Suppose that g is not continuous at $u^* \in V$. Then, since

$$\lim_{u \to u^*-0} g(u) < \lim_{u \to u^*+0} g(u)$$

and g is non-decreasing, there is no $u' \in V$ such that

$$g(u') \in \left(\lim_{u \to u^*-0} g(u), \lim_{u \to u^*+0} g(u) \right).$$

On the other hand, as shown in Step 5 of Lemma 1, $\inf \{G(\varphi)|\varphi \in \Phi^+ \setminus \{id_V\}\} = 0$, so there is $\varphi' \in \Phi$ such that

$$G(\varphi') \in \left(\lim_{u \to u^*-0} g(u), \lim_{u \to u^*+0} g(u) \right).$$

For $\varphi'(u_0) \in V$, we have

$$\begin{aligned} g(\varphi'(u_0)) &= \sup\{G(\varphi)|\varphi(u_0) \leq \varphi'(u_0)\} \\ &= G(\varphi') \in \left(\lim_{u \to u^*-0} g(u), \lim_{u \to u^*+0} g(u)\right), \end{aligned}$$

which is a contradiction.■

Step 3. *For each $u \in V$ and each $\varphi \in \Phi$, $g(\varphi(u)) = g(u) + G(\varphi)$.*
Proof. For each $u \in V$ and each $\varphi \in \Phi$,

$$\begin{aligned} g(\varphi(u)) &= \sup\{G(\varphi')|\varphi'(u_0) \leq \varphi(u)\} \\ &= \sup\{G(\varphi'' \circ \varphi)|\varphi'' \circ \varphi(u_0) \leq \varphi(u)\} \\ &= \sup\{G(\varphi'') + G(\varphi)|\varphi''(u_0) \leq u\} \\ &= G(\varphi) + \sup\{G(\varphi'')|\varphi''(u_0) \leq u\} \\ &= G(\varphi) + g(u).■ \end{aligned}$$

If g is increasing, then the transformed utilitarian rule associated with g satisfies SP and IIA. However, it is not obvious whether g is increasing, so we need to consider the case where g is not increasing.

We introduce the following notation. For each $x \in \mathbb{R}$, let $g^{-1}(x) \equiv \{u' \in V|g(u') = x\}$. Since g is non-decreasing and continuous, for each $x \in \mathbb{R}$, $g^{-1}(x)$ is a closed interval. Let $S \equiv \{u \in V|\exists u' \in V,\ u' \neq u,\ g(u') = g(u)\}$. Clearly, g is increasing if and only if S is empty.

Step 4. *For each $\varphi \in \Phi$ and each $x \in \mathbb{R}$, if $g^{-1}(x) = [\widetilde{u}, \overline{u}]$, then $g^{-1}(x + G(\varphi)) = [\varphi(\widetilde{u}), \varphi(\overline{u})]$.*
Proof. For each $x \in \mathbb{R}$, let $\widetilde{u} = \arg\min g^{-1}(x)$ and $\overline{u} = \arg\min g^{-1}(x)$. Then, by

Step 3, for each $\varphi \in \Phi$,

$$\begin{aligned} g(\varphi(\widetilde{u})) &= G(\varphi) + g(\widetilde{u}) \\ &= G(\varphi) + x \\ &= G(\varphi) + g(\overline{u}) \\ &= g(\varphi(\overline{u})) \end{aligned}$$

Hence, $g^{-1}(x + G(\varphi)) \supseteq [\varphi(\widetilde{u}), \varphi(\overline{u})]$.

For each $u' < \varphi(\widetilde{u})$, since $\varphi^{-1}(u') < \widetilde{u}$,

$$\begin{aligned} g(u') &= G(\varphi) + g(\varphi^{-1}(u')) \\ &< G(\varphi) + g(\widetilde{u}). \end{aligned}$$

Hence, $u' \notin g^{-1}(x + G(\varphi))$. We can similarly show that for each $u' > \varphi(\overline{u})$, $u' \notin g^{-1}(x + G(\varphi))$.■

Define the binary relation $\approx$ over S such that for each pair $u, u' \in S$,

$$u \approx u' \text{ if and only if there is } \varphi \in \Phi \text{ such that } u' = \varphi(u).$$

By Step 4, for each $u \in S$ and each $\varphi \in \Phi$, $\varphi(u) \in S$, so that $\approx$ is an equivalence relation. Let $S/\approx$ be the quotient set of S by $\approx$. An element of $S/\approx$ is denoted by $[u_\lambda]$, where $u_\lambda \in S$ denotes a representative. We denote that for each pair $u_\lambda, u_{\lambda'} \in S$, if u_λ is a representative and $g(u_\lambda) = g(u_{\lambda'})$, then $u_{\lambda'}$ is a representative. Define $g_2 : V \to [0,1]$ by

$$g_2(u) = \begin{cases} \frac{u_\lambda - \min g^{-1}(g(u_\lambda))}{\max g^{-1}(g(u_\lambda)) - \min g^{-1}(g(u_\lambda))}, & \text{if } u \in [u_\lambda] \\ 0, & \text{if } u \notin S. \end{cases}$$

Step 5. *For each $\varphi \in \Phi$ and each $u \in V$, $g_2(\varphi(u)) = g_2(u)$.*

Proof. By Step 4, for each $u \in V$, if $u \notin S$, then $\varphi(u) \notin S$. Hence,

$$g_2(\varphi(u)) = g_2(u) = 0.$$

For each $u \in V$, if $u \in [u_\lambda]$, then there is $\varphi' \in \Phi$ such that $\varphi'(u) = u_\lambda$. Then, for each $\varphi \in \Phi$,

$$\varphi' \circ \varphi^{-1}(\varphi(u)) = \varphi'(u) = u_\lambda,$$

so that $\varphi(u) \in [u_\lambda]$. Hence,

$$\begin{aligned} g_2(\varphi(u)) &= \frac{u_\lambda - \min g^{-1}(g(u_\lambda))}{\max g^{-1}(g(u_\lambda)) - \min g^{-1}(g(u_\lambda))} \\ &= g_2(u). \blacksquare \end{aligned}$$

Step 6. *Let R be a* SWFL *such that for each $U \in \mathcal{U}^N$ and each pair $x, y \in X$, $x\, P_U\, y$ if and only if* (1) $\sum_{i \in N} g(U_i(x)) > \sum_{i \in N} g(U_i(y))$ *or* (2) $\sum_{i \in N} g(U_i(x)) = \sum_{i \in N} g(U_i(y))$ *and* $\sum_{i \in N} g_2(U_i(x)) > \sum_{i \in N} g_2(U_i(y))$. *Then, R satisfies* SP, AN, IIA, *and* INV-Φ.

Proof. Obviously, R satisfies AN and IIA. To prove that R satisfies SP, it suffices to show that for each pair $u, u' \in V$ such that $u < u'$, either (1) $g(u) < g(u')$ or (2) $g(u) = g(u')$ and $g_2(u) < g_2(u')$ holds. Since g is non-decreasing, either $g(u) < g(u')$ or $g(u) = g(u')$ holds. In addition, if $g(u) = g(u')$, then there are $[u_\lambda]$, $[u_{\lambda'}]$ and $\varphi \in \Phi$ such that $u_\lambda < u_{\lambda'}$, $\varphi(u) = u_\lambda$ and $\varphi(u') = u_{\lambda'}$. Hence,

$$\begin{aligned} g_2(u) &= \frac{u_\lambda - \min g^{-1}(g(u_\lambda))}{\max g^{-1}(g(u_\lambda)) - \min g^{-1}(g(u_\lambda))} \\ &< \frac{u_{\lambda'} - \min g^{-1}(u_{\lambda'})}{\max g^{-1}(g(u_\lambda)) - \min g^{-1}(g(u_\lambda))} \\ &= g_2(u). \end{aligned}$$

Next, we show that R satisfies INV-Φ. For each pair $x, y \in X$, each $U \in \mathcal{U}^N$, each $i \in N$, and each $\varphi \in \Phi$, by Step 3, $\sum_{i \in N} g(U_i(x)) \geq \sum_{i \in N} g(U_i(y))$ if and only if $\sum_{j \neq i} g(U_j(x)) + g(\varphi \circ U_i(x)) \geq \sum_{j \neq i} g(U_j(y)) + g(\varphi \circ U_i(y))$. Also, by Step 5, $\sum_{i \in N} g_2(U_i(x)) \geq \sum_{i \in N} g_2(U_i(y))$ if and only if $\sum_{j \neq i} g_2(U_j(x)) + g_2(\varphi \circ U_i(x)) \geq$

$\sum_{j\neq i} g_2(U_j(y)) + g_2(\varphi \circ U_i(y))$.

Hence, $x\, R_U\, y$, if and only if $x\, R_{(\varphi\circ U_i, U_{-i})}\, y$.■

6.4 Proof of Theorem 2

Suppose that there are distinct R and R' that satisfy SP, AN, IIA and INV-Φ. By the welfarism theorem, the SWOs $\succsim_R$ and $\succsim_{R'}$, generated by R and R', respectively, are distinct orderings. Then, there are $(u_1, u_2, \cdots, u_n)$ and $(u'_1, u'_2, \cdots, u'_n) \in V^N$ such that $(u_1, u_2, \cdots, u_n) \succsim_R (u'_1, u'_2, \cdots, u'_n)$ and $(u'_1, u'_2, \cdots, u'_n) \succ_{R'} (u_1, u_2, \cdots, u_n)$. Let $u_0 \in V$. By RC of Φ, for each $i \in N$, there are $\varphi_i, \psi_i \in \Phi$ such that $\varphi_i(u_0) = u_i$ and $\psi_i(u_0) = u'_i$. Then, by INV-Φ and AN, we can show

$$(u_1, u_2, \cdots, u_n) \sim_R (u_0, \varphi_1 \circ \varphi_2(u_0), \cdots, u_n),$$

in the similar way with the proof of the necessity part of Theorem 1. Taking similar steps, we obtain

$$\begin{aligned}
(u_1, u_2, \cdots, u_n) &\sim_R (u_0, u_0, \cdots, \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n(u_0)),\\
(u'_1, u'_2, \cdots, u'_n) &\sim_R (u_0, u_0, \cdots, \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n(u_0)),\\
(u_1, u_2, \cdots, u_n) &\sim_{R'} (u_0, u_0, \cdots, \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n(u_0)), \text{ and}\\
(u'_1, u'_2, \cdots, u'_n) &\sim_{R'} (u_0, u_0, \cdots, \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n(u_0)).
\end{aligned}$$

Hence,

$$(u_0, u_0, \cdots, \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n(u_0)) \succsim_R (u_0, u_0, \cdots, \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n(u_0)),$$

which implies by SP

$$\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n(u_0) \geq \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n(u_0).$$

Similarly,

$$\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n(u_0) < \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n(u_0),$$

which is a contradiction. Hence, there is at most one SWFL.■

6.5 Proof of Proposition 3

Step 1. If the utilitarian rule satisfies INV-Φ^N then $\Phi^N \subseteq \prod_{i \in N} \Phi^{PAT}$.

Proof. Suppose there is $\varphi \in \Phi^N$ such that $\varphi \notin \prod_{i \in N} \Phi^{PAT}$. Then, there is $i \in N$ such that $\varphi_i \notin \Phi^{PAT}$. Then, by Basu's Theorem (Basu [27]), there are $[\underline{u}, \overline{u}], [\underline{u}', \overline{u}'] \subseteq V$ such that $\overline{u} - \underline{u} = \overline{u}' - \underline{u}'$ and $\varphi_i(\overline{u}) - \varphi_i(\underline{u}) \neq \varphi_i(\overline{u}') - \varphi_i(\underline{u}')$. Let $U, U' \in \mathcal{U}^N$ be such that for $i \in N$, $U_i(x) = \overline{u}$, $U_i(y) = \underline{u}$, $U_i'(x) = \overline{u}'$, and $U_i'(y) = \underline{u}'$; for $j \neq i$, $U_i(x) = U_i'(x) = \underline{u}$ and $U_i(y) = U_i'(y) = \overline{u}$; and for each $k \in N \backslash \{i, j\}$, $U_k(x) = U_k(y)$ and $U_k'(x) = U_k'(y)$. Since $\sum_{i \in N} U_i(x) = \sum_{i \in N} U_i(y)$ and $\sum_{i \in N} U_i'(x) = \sum_{i \in N} U_i'(y)$, the utilitarian rule R concludes $xI_U y$ and $xI_{U'} y$.

By INV-Φ^N, we have $xI_{\varphi \circ U} y$ and $xI_{\varphi \circ T'} y$. $xI_{\varphi \circ U} y$ implies $\sum_{i \in N} \varphi_i \circ U_i(x) = \sum_{i \in N} \varphi_i \circ U_i(y)$, so that

$$\varphi_i(\overline{u}) - \varphi_i(\underline{u}) = \varphi_j(\overline{u}) - \varphi_j(\underline{u}).$$

Similarly, $xI_{\varphi \circ U} y$ implies $\sum_{i \in N} \varphi_i \circ U_i'(x) = \sum_{i \in N} \varphi_i \circ U_i'(y)$, so that

$$\varphi_i(\overline{u}') - \varphi_i(\underline{u}') = \varphi_j(\overline{u}) - \varphi_j(\underline{u}),$$

which is a contradiction.■

Step 2. If the utilitarian rule satisfies INV-Φ^N then $\Phi^N \subseteq \Phi^{CUC}$.

Proof. Suppose there is $\varphi \in \Phi^N$ such that $\varphi \notin \Phi^{CUC}$. Since $\varphi \in \prod_{i \in N} \Phi^{PAT}$ by Step 1, there are $i, j \in N, i \neq j$ such that for each $u \in V$, $\varphi_i(u) = a_i u + b_i$, $\varphi_j(u) = a_j u + b_j$ and $a_i \neq a_j$. Let $U \in \mathcal{U}^N$ be such that for $i \in N$, $U_i(x) = 2$, $U_i(y) = 1$; for $j \in N$, $U_j(x) = 1$ and $U_i(y) = 2$; and for each $k \in N \backslash \{i, j\}$, $U_k(x) = U_k(y) = 1$. Since $\sum_{i \in N} U_i(x) = \sum_{i \in N} U_i(y)$, the utilitarian rule R concludes $xI_U y$.

By INV-Φ^N, we have $xI_{\varphi \circ U} y$. $xI_{\varphi \circ U} y$ implies $\sum_{i \in N} \varphi_i \circ U_i(x) = \sum_{i \in N} \varphi_i \circ U_i(y)$, so that

$$\varphi_i \circ U_i(x) - \varphi_i \circ U_i(y) = \varphi_j \circ U_j(y) - \varphi_j \circ U_j(x).$$

This equation implies $a_i = a_j$, which is a contradiction.■

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