Weighted values and the Core in NTU games

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Abstract

The purpose of this paper is to extend the following result by Monderer et al. (1992): the set of weighted Shapley values includes the Core in TU games. We extend the result to the class of uniformly positively smooth NTU games. We focus on two solution concepts which extend the weighted Shapley value. First, we focus on the weighted Egalitarian solution introduced by Kalai and Samet (1985). We show that the set of weighted Egalitarian solutions includes the Core. Second, we focus on a new solution concept which we call the consistent weighted Shapley value. The solution concept is defined by extending the consistent Shapley value by Maschler and Owen (1989). We show that, if the attainable payoff for grand coalition is represented by a closed half-space of a hyperplane, then the set of consistent weighted Shapley values includes the Core.

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1 Introduction

In the class of TU cooperative games, there are two major solution concepts: the Core and the weighted Shapley value by Shapley (1953). With respect to the two solution concepts, Monderer et al. (1992) proved that the set of weighted Shapley values includes the Core. This means that, given an arbitrary element of the Core, we can always find a weight with which the weighted Shapley value coincides with the element. The purpose of this

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paper is to extend the result to the class of uniformly positively smooth NTU games.

In order to extend the result, we need to extend the weighted Shapley value to NTU games. We consider two extensions. First, we focus on the weighted Egalitarian solution by Kalai and Samet (1985). Theorem 1 of this paper shows that the set of weighted Egalitarian solutions includes the Core. Second, we focus on the consistent Shapley value by Maschler and Owen (1989). We extend the value by incorporating positive weights in its definition and define a new solution concept, the consistent weighted Shapley value. The value determines payoff vector by calculating the expected value of marginal contributions. Theorem 2 of this paper shows that, if the attainable payoff for grand coalition is represented by a closed half-space of a hyperplane, then the set of consistent weighted Shapley values includes the Core.

The outline of the proof of theorems is the following. We first consider an arbitrary solution function which incorporates positive weights in its definition. We call such a solution function a weighted solution function. We show that, if a weighted solution function satisfies three conditions, then the closure of the range of the weighted solution function includes the Core. Then, we prove that the weighted Egalitarian solution and the consistent weighted Shapley value satisfy the three conditions.

This paper is organized as follows. Section 2 is preliminary. In Section 3, we discuss the relationship between the Core and a weighted solution function. In Section 4, we discuss the relationship between the Core and the weighted Egalitarian solution. In Section 5, we discuss the relationship between the Core and the consistent weighted Shapley value. Section 6 gives concluding remarks and an example.

2 Preliminary

Let N denote a set of players such that $N = \{1, \cdots, n\}$. For each non-empty $S \subseteq N$, let $\mathbb{R}^S$ denote the $|S|$-dimensional Euclidean space. $\mathbb{R}^S_+$ denotes the space with non-negative coordinates, and $\mathbb{R}^S_{++}$ denotes the space with positive coordinates. For each $x, y \in \mathbb{R}^N$, we define $\min\{x, y\}$ by

$$\min\{x, y\} = (z_i, \cdots, z_n) \in \mathbb{R}^N, \text{ where } z_k = \min\{x_k, y_k\} \text{ for all } k = 1, \cdots, n.$$

We define $\max\{x, y\}$ in a parallel manner. We define vector inequalities as follows: for each non-empty $S \subseteq N$ and $x, y \in \mathbb{R}^S$, $x \gg y$ means $x_i > y_i$ for all $i \in S$; $x \geq y$ means $x_i \geq y_i$ for all $i \in S$; $x > y$ means $x \geq y$ and $x \neq y$.

For each $x \in \mathbb{R}^N$ and non-empty $S \subset N$, let $x_S \in \mathbb{R}^S$ denote the projection of x on S, i.e., $(x_S)_i = x_i$ for all $i \in S$. For a subset X of $\mathbb{R}^S$, let ∂X denote the boundary of X, and let clX denote the closure of X. For each $x, y \in \mathbb{R}^S$, let $x \cdot y$ denote the inner product. For each non-empty $S \subseteq N$, we define

$$\Delta^S_{++} = \Big\{x \in \mathbb{R}^S : x_i > 0 \text{ for all } i \in S \text{ and } \sum_{i \in S} x_i = 1\Big\}.$$

A NTU game is a pair (N, V) where V is a function which associates with each non-empty coalition $S \subseteq N$ a subset of $\mathbb{R}^S$. Here, $x \in V(S)$ means that the payoff vector x is attainable by cooperation of players in S. We make assumptions on V.[1] for each non-empty $S \subseteq N$, $V(S)$ is

N1: a non-empty proper subset of $\mathbb{R}^S$.

N2: closed, convex and comprehensive, i.e., $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$.

N3: uniformly positively smooth; at each $x \in \partial V(S)$, there exists a unique $\lambda(x) \in \Delta^S_{++}$ such that $V(S) \subseteq \{y \in \mathbb{R}^S : \lambda(x) \cdot y \leq \lambda(x) \cdot x\}$. Moreover, there exists $\delta \in \mathbb{R}^S_{++}$ such that $\lambda(x) \geq \delta$ for all $x \in \partial V(S)$.

Let $\mathcal{G}$ denote the set of NTU games which satisfy N1 to N3. In the remaining part, we fix player set N. So, we write V instead of (N, V).

We review the definition of the Core. Let $V \in \mathcal{G}$ and $x, y \in V(N)$. We say that y dominates x via a coalition S if $y_i > x_i$ for all $i \in S$ and $y_S \in V(S)$. We say that y dominates x if there exists a coalition S such that y dominates x via S. We define the core of V by

$$C(V) = \{x \in \mathbb{R}^N : \text{there does not exist a vector } y \in \mathbb{R}^N \text{ which dominates } x\}.$$

3 The Core and weighted solution functions

In this section, we consider a general solution function which incorporates positive weights in its definition. We start with the interpretation of using positive weights in the definition of a solution function.

The Shapley value was extended to the weighted Shapley value by incorporating positive weights. One interpretation of using positive weights is to capture asymmetric importance of players. Hart and Mas-Colell (1989) gave an example of cost allocation problem among investment projects. In

[1] We follow the assumptions of Hart (2005).

this problem, the characteristic function describes the cost of implementing the projects, while it does not capture the profitability from the different projects. We can associate with each project a positive number w_i which captures the profitability of project i. If we need to assign an asymmetric payoff depending on the importance, then using a positive weight is a useful way.

We now combine positive weights and a solution function in the class of NTU games. We call a function ψ which has the form $\psi : \mathcal{G} \times \Delta^N_{++} \to \mathbb{R}^N$ a weighted solution function. Let $V \in \mathcal{G}$ be fixed and we restrict the domain of ψ to Δ^N_{++}, i.e., we consider a function $\psi^w(V) : \Delta^N_{++} \to \mathbb{R}^N$. In the remaining part, we assume the following three conditions on $\psi^w(V)$:

C1: $\psi^w(V) \in \partial V(N)$ for all $w \in \Delta^N_{++}$.

C2: $\psi^w(V)$ is continuous with respect to w.

C3: Let $\{w^k\}_{k=1}^{\infty}$ be a convergent sequence such that there exists a non-empty coalition $T \subset N$ which satisfies $\lim_{k\to\infty} w_i^k = 0$ for all $i \in T$ and $\lim_{k\to\infty} w_j^k > 0$ for all $j \in N \backslash T$. Then, $\{\psi^{w^k}(V)\}_{k=1}^{\infty}$ has a convergent subsequence which satisfies

$$\lim_{k\to\infty} \psi_T^{w^k}(V) \in V(T).$$

Based on the interpretation that positive weights represent the importance of players, we examine the meaning of each condition. First, C1 is a standard requirement. The condition says that the weighted solution function always prescribes a pareto optimal outcome. C2 says that if w slightly changes, then the outcome also slightly changes. A slight change in w means a slight change in importance of players. Then, it seems natural to conclude that the outcome also slightly changes. Finally, we examine the meaning of C3. The condition starts from the consideration of extreme case, i.e., there is a set of players T whose weights go to 0. What is intended here is that the importance of players in T becomes smaller and smaller. In this case, C3 requires that the payoff of players in T is determined separately from the payoff of players in $N \backslash T$. Namely, players in T just receive what they can obtain by their own.

The main result of this section is the following: if a weighted solution function satisfies the three conditions, then any element of the Core is attainable as the outcome of the function.

Proposition 1 *Let $V \in \mathcal{G}$. If a weighted solution function $\psi^w(V) : \Delta^N_{++} \to$*

$\mathbb{R}^N$ *satisfies C1 to C3, then*

$$C(V) \subseteq cl\{\psi^w(V) : w \in \Delta^N_{++}\}.$$

Before proving this result, we give one remark on the assumption N3. In view of proving Proposition 1, the following weaker condition of N3 is sufficient: for each non-empty $S \subseteq N$,

N3': $x, y \in \partial V(S)$ and $x \leq y$ imply $x = y$.

We say that $V(S)$ is non-levelled if it satisfies the condition N3'.

Proof of Proposition 1. Suppose that $C(V) \neq \emptyset$ and let $x \in C(V)$. Let $\Psi : \Delta^N_{++} \to \mathbb{R}^N$ denote the following function:[2]

$$\Psi(w) = x - \psi^w(V). \tag{1}$$

Since $\psi^w(V)$ is continuous from C2, Ψ is also continuous. For any $w \in \Delta^N_{++}$, let $\tilde{\Psi}(w) = \min\{\Psi(w), \mathbf{1}\}$. For any $\epsilon \in (0,1)$, we define

$$S_\epsilon = \left\{ w \in \Delta^N_{++} : w_i \geq \frac{\epsilon}{1+2n} \text{ for all } i \in N \right\}.$$

The set is compact and convex. We can also check that the set is non-empty; for any $\epsilon \in (0,1)$, let $d \in \Delta^N_{++}$ denote the following vector:

$$d_i = 2 + \frac{1}{n} \Big/ 1 + 2n \text{ for all } i = 1, \cdots, n.$$

Then, $d \in S_\epsilon$.

We define a function $g : S_\epsilon \to \mathbb{R}^N$ as follows:

$$g_i(w) = \frac{\epsilon + w_i + \max\{0, \tilde{\Psi}_i(w)\}}{n\epsilon + 1 + \sum_{j \in N} \max\{0, \tilde{\Psi}_j(w)\}} \text{ for all } i \in N. \tag{2}$$

Note that

$$g_i(w) \geq \frac{\epsilon}{n\epsilon + 1 + n} \geq \frac{\epsilon}{1+2n} \text{ for all } i \in N.$$

As a result, $g : S_\epsilon \to S_\epsilon$ is a continuous function from the compact, convex and non-empty set to itself. From Brouwer's fixed point theorem, there exists a fixed point. For any $k \in \mathbb{N}$, $k \geq 2$, let $w^{\frac{1}{k}} \in S_{\frac{1}{k}}$ denote the vector which satisfies $g(w^{\frac{1}{k}}) = w^{\frac{1}{k}}$.

[2] In the construction of this function, we mimic the proof of Theorem 5.5 by Jehle and Reny (2011).

Now, consider the sequence $\{w^{\frac{1}{k}}\}_{k=2}^{\infty}$. Since the sequence is bounded, there exists a convergent subsequence. Take any convergent subsequence $\{w^{\frac{1}{k}}\} \subseteq \{w^{\frac{1}{k}}\}_{k=2}^{\infty}$. From equation (2), we have, for any k,

$$w_i^{\frac{1}{k}} \left[\frac{n}{k} + \sum_{j \in N} \max\{0, \tilde{\Psi}_j(w^{\frac{1}{k}})\} \right] = \frac{1}{k} + \max\{0, \tilde{\Psi}_i(w^{\frac{1}{k}})\} \text{ for all } i \in N. \quad (3)$$

Let w^* be the limit point of $\{w^{\frac{1}{k}}\}$, that is, $w^{\frac{1}{k}} \to w^*$. Since $\sum_{i \in N} w_i^* = 1$ and $w_i^* \geq 0$, $i = 1, \cdots, n$, there is at least one player i such that $w_i^* > 0$. Without loss of generality, assume that

$$\begin{aligned} w_i^* &> 0 \text{ for } i = 1, \cdots, t, \\ w_j^* &= 0 \text{ for } j = t+1, \cdots, n. \end{aligned}$$

We define $S = \{1, \cdots, t\}$ and $T = \{t+1, \cdots, n\}$. Note that T might be empty.

Case 1: If $S = N$, we have $w_i^* > 0$ for all $i \in N$. From C2, $\lim_{k \to \infty} \tilde{\Psi}_i(w^{\frac{1}{k}}) = \tilde{\Psi}_i(w^*)$ for all $i \in N$. So, by taking the limit $k \to \infty$ of both sides of equation (3)

$$w_i^* \left[\sum_{j \in N} \max\{0, \tilde{\Psi}_j(w^*)\} \right] = \max\{0, \tilde{\Psi}_i(w^*)\} \text{ for all } i \in N. \quad (4)$$

Assume that $\left[\sum_{j \in N} \max\{0, \tilde{\Psi}_j^*\} \right] > 0$. Then, from equation (4), $\tilde{\Psi}_i(w^*) > 0$ for all $i \in N$. In this case, there exists a sufficiently large k' such that $\tilde{\Psi}_i(w^{\frac{1}{k'}}) > 0$ for all $i \in N$. It follows that

$$\begin{aligned} \tilde{\Psi}_i(w^{\frac{1}{k'}}) = \min\{\Psi_i(w^{\frac{1}{k'}}), 1\} &> 0 \text{ for all } i \in N, \\ \Psi_i(w^{\frac{1}{k'}}) &> 0 \text{ for all } i \in N, \\ x_i &> \psi_i^{w^{\frac{1}{k'}}}(V) \text{ for all } i \in N. \end{aligned}$$

Since $x \in C(V)$, we have $x \in \partial V(N)$. From C1, $\psi^{w^{\frac{1}{k'}}}(V) \in \partial V(N)$. From N3', $x = \psi(w^{\frac{1}{k'}})$, which contradicts $x_i > \psi_i(w^{\frac{1}{k'}})$ for all $i \in N$.

As a result, we must have $\left[\sum_{j \in N} \max\{0, \tilde{\Psi}_j^*\} \right] = 0$. From equation (4),

$$\begin{aligned} \tilde{\Psi}_i(w^*) &\leq 0 \text{ for all } i \in N, \\ \lim_{k \to \infty} \min\{\Psi_i(w^{\frac{1}{k}}), 1\} &\leq 0 \text{ for all } i \in N, \\ \lim_{k \to \infty} \Psi_i(w^{\frac{1}{k}}) &\leq 0 \text{ for all } i \in N. \end{aligned}$$

From equation (1),

$$\lim_{k\to\infty}\Big\{x_i - \psi_i^{w^{\frac{1}{k}}}(V)\Big\} \le 0 \text{ for all } i \in N,$$

$$x_i \le \psi_i^{w^*}(V) \text{ for all } i \in N.$$

Since $x, \psi^{w^*}(V) \in \partial V(N)$, from N3', $x = \psi^{w^*}(V)$. It follows that $x = \lim_{k\to\infty} \psi^{w^{\frac{1}{k}}}(V)$.

Case 2: The remaining possibility is that $1 \le |S| < n$. Consider the sequence $\psi^{w^{\frac{1}{k}}}(V)$. From C3, $\psi^{w^{\frac{1}{k}}}(V)$ has a convergent subsequence. In order to simplify the notation, suppose that $\psi^{w^{\frac{1}{k}}}(V)$ itself converges and let ψ^* denote the limit point. Since $\psi^{w^{\frac{1}{k}}}(V)$ converges, $\Psi(w^{\frac{1}{k}})$ and $\tilde{\Psi}(w^{\frac{1}{k}}) = \min\{\Psi(w^{\frac{1}{k}}), \mathbf{1}\}$ also converge. Let $\tilde{\Psi}^*$ denote the limit point of $\tilde{\Psi}(w^{\frac{1}{k}})$. By taking the limit $k \to \infty$ of both sides of equation (3),

$$w_i^*\Big[\sum_{j\in N} \max\big\{0, \tilde{\Psi}_j^*\big\}\Big] = \max\big\{0, \tilde{\Psi}_i^*\big\} \text{ for all } i \in N. \tag{5}$$

Assume that $\Big[\sum_{j\in N} \max\big\{0, \tilde{\Psi}_j^*\big\}\Big] > 0$. Then, from equation (5),

$$\begin{cases} \tilde{\Psi}_i^* > 0 & \text{for all } i \in S, \\ \tilde{\Psi}_j^* \le 0 & \text{for all } j \in T. \end{cases}$$

From the definition of $\tilde{\Psi}_i^*$, we have

$$\begin{cases} \tilde{\Psi}_i^* = \lim_{k\to\infty} \tilde{\Psi}_i(w^{\frac{1}{k}}) = \lim_{k\to\infty} \min\{\Psi_i(w^{\frac{1}{k}}), 1\} > 0 & \text{for all } i \in S, \\ \tilde{\Psi}_j^* = \lim_{k\to\infty} \tilde{\Psi}_j(w^{\frac{1}{k}}) = \lim_{k\to\infty} \min\{\Psi_j(w^{\frac{1}{k}}), 1\} \le 0 & \text{for all } j \in T. \end{cases}$$

The above two conditions imply

$$\begin{cases} \lim_{k\to\infty} \Psi_i(w^{\frac{1}{k}}) > 0 & \text{for all } i \in S, \\ \lim_{k\to\infty} \Psi_j(w^{\frac{1}{k}}) \le 0 & \text{for all } j \in T. \end{cases}$$

Let us focus on the sequence $\Psi_j(w^{\frac{1}{k}})$ for $j \in T$. Since $\lim_{k\to\infty} \Psi_j(w^{\frac{1}{k}}) \le 0$,

$$x_j - \lim_{k\to\infty} \psi_j^{w^{\frac{1}{k}}}(V) \le 0 \text{ for all } j \in T,$$

$$x_j \le \psi_j^* \text{ for all } j \in T.$$

From C3, $\psi_T^* \in V(T)$. Since $x \in C(V)$, we have $\psi_T^* \in \partial V(T)$. From N3', $x_j = \psi_j^*$ for all $j \in T$. On the other hand, for each player $i \in S$, we have

$$\lim_{k\to\infty} \Psi_i(w^{\frac{1}{k}}) > 0 \text{ for all } i \in S,$$

$$x_i > \psi_i^* \text{ for all } i \in S.$$

It follows that $x \geq \psi^*$. Since $x, \psi^* \in \partial V(N)$, from N3', $x = \psi^*$, which contradicts $x_i > \psi_i^*$ for all $i \in S$.

As a result, we must have $\left[\sum_{i \in N} \max\{0, \tilde{\Psi}_i^*\}\right] = 0$. From equation (5),

$$\tilde{\Psi}_i^* \leq 0 \text{ for all } i \in N,$$
$$\lim_{k \to \infty} \min\{\Psi_i(w^{\frac{1}{k}}), 1\} \leq 0 \text{ for all } i \in N,$$
$$\lim_{k \to \infty} \Psi_i(w^{\frac{1}{k}}) \leq 0 \text{ for all } i \in N.$$

From equation (1),

$$\lim_{k \to \infty} \left\{x_i - \psi_i^{w^{\frac{1}{k}}}(V)\right\} \leq 0 \text{ for all } i \in N,$$
$$x_i \leq \psi_i^* \text{ for all } i \in N.$$

Since $x, \psi^* \in \partial V(N)$, from N3', $x = \psi^*$. Thus, we have $x = \lim_{k \to \infty} \psi^{w^{\frac{1}{k}}}(V)$, which completes the proof. □

4 Weighted Egalitarian solution

In this section, we prove that the set of weighted Egalitarian solutions includes the Core by using the result of Section 3. We define the solution function by following the notation of Kalai and Samet (1985). Let $V \in \mathcal{G}$ and $w \in \Delta_{++}^N$. We define $D^w(V, S) \in \mathbb{R}^S$ and $Z^w(V, S) \in \mathbb{R}^S$ for non-empty $S \subseteq N$ inductively as follows: for each $i \in N$,

$$Z^w(V, \{i\}) = D^w(V, \{i\}) = \max\{x \in \mathbb{R} : x \in V(\{i\})\}.$$

For each $S \subseteq N$, $|S| \geq 2$,

$$Z_i^w(V, S) = \sum_{T \subset S : i \in T} D_i^w(V, T) \text{ for all } i \in S.$$
$$D_i^w(V, S) = w_i \max\{t : (Z^w(V, S) + t w_S) \in V(S)\} \text{ for all } i \in S. \quad (6)$$

Note that, in equation (6), there always exists a real number t which attains the maximum from N1 (proper subset) and N2 (closed, comprehensive). In addition, from N3', such a real number t is always unique. Let us introduce a new function which assigns the unique real number t to each $w \in \Delta_{++}^N$ and $S \subseteq N$, $S \neq \emptyset$. Formally, let $Q : \Delta_{++}^N \times 2^N \backslash \emptyset \to \mathbb{R}$ denote the function which satisfies

$$D_i^w(V, S) = w_i Q(w, S) \text{ for all } w \in \Delta_{++}^N, S \subseteq N, S \neq \emptyset.$$

We define the weighted Egalitarian solution $\xi^w : \mathcal{G} \to \mathbb{R}^N$ with positive weight w as follows:

$$\xi_i^w(V) = \sum_{S \subseteq N : i \in S} D_i^w(V, S) \text{ for all } i \in N, V \in \mathcal{G}.$$

Let $V \in \mathcal{G}$. We define the set of weighted Egalitarian solutions $\Xi(V)$ as follows:

$$\Xi(V) = \{\xi^w(V) : w \in \Delta_{++}^N\}.$$

On the class of TU games, ξ^w coincides with the weighted Shapley value with positive weight w; see Theorem 4 of Kalai and Samet (1985).

We prove that, for any $V \in \mathcal{G}$, $\xi^w(V)$ satisfies the conditions C1 to C3. In order to do that, it is helpful to show that the range of the function $\xi^w(V) : \Delta_{++}^N \to \mathbb{R}^N$ is bounded. Let us first consider the following two conditions on $V(S)$, where $S \subseteq N$, $S \neq \emptyset$:

N4: $\{x^k\}_{k=1}^\infty \subseteq \partial V(S)$ and $x_i^k \to +\infty$ for some $i \in S$ implies $x_j \to -\infty$ for some $j \in S$.

N5: $\{x^k\}_{k=1}^\infty \subseteq \partial V(S)$ and $x_i^k \to -\infty$ for some $i \in S$ implies $x_j \to +\infty$ for some $j \in S$.

Lemma 1 *$V \in \mathcal{G}$ satisfies N4.*

Proof. Let $x \in \partial V(S)$. Then, from N3, there exists a unique $\lambda(x)$ such that

$$V(S) \subseteq \{y \in \mathbb{R}^S : y \cdot \lambda(x) \leq x \cdot \lambda(x)\}.$$

Consider a sequence $\{x^k\}_{k=1}^\infty \subseteq \partial V(S)$ such that $x_i^k \to +\infty$ for some $i \in N$. Then, $x^k \cdot \lambda(x) \leq x \cdot \lambda(x)$ for all $k = 1, 2, \cdots$. Since $\lambda(x) \in \Delta_{++}^N$ and $x_i^k \to +\infty$, $x_i^k \cdot \lambda_i(x) \to +\infty$. Since the sequence $x^k \cdot \lambda(x)$ is bounded from above, there must be a player $j \in N$ such that $x_j^k \to -\infty$. $\square$

Lemma 2 *$V \in \mathcal{G}$ satisfies N5.*

Proof. Consider a sequence $\{x^k\}_{k=1}^\infty \subseteq \partial V(S)$ such that $x_i^k \to -\infty$ for some $i \in N$. From N3, for each x^k, there exists a unique $\lambda(x^k)$ such that

$$V(S) \subseteq \{y \in \mathbb{R}^S : y \cdot \lambda(x^k) \leq x^k \cdot \lambda(x^k)\}.$$

Let $x \in V(S)$ be given. Then, $x \cdot \lambda(x^k) \leq x^k \cdot \lambda(x^k)$ for all $k = 1, 2, \cdots$. Again, from N3, there exists $\delta \in \mathbb{R}_{++}^S$ such that $\lambda(x^k) \geq \delta$ for all $k = 1, 2, \cdots$. It follows that $x_i^k \cdot \lambda_i(x^k) \to -\infty$. Since the sequence $x^k \cdot \lambda(x^k)$ is bounded from below, there must be a player $j \in N$ such that $x_j^k \to +\infty$. $\square$

Remark 1 Even if we replace N3 with the three conditions N3', N4 and N5, all results discussed in this paper remain valid. Since N3 is a standard condition in the literature, we apply the condition. ∎

The above two lemmas imply the following lemma:

Lemma 3 *Let $V \in \mathcal{G}$, $i \in N$ and $S \subseteq N$, $i \in S$. Then, there exists $M > 0$ and $m < 0$ such that for all $w \in \Delta^N_{++}$, $m \leq D^w_i(V,S) \leq M$.*

Proof. We proceed by induction. If $S = \{i\}$, then for all $w \in \Delta^N_{++}$, $D^w_i(V,\{i\}) = \max\{x_i : x \in V(\{i\})\}$ and the statement holds. Suppose that the result holds for $T \subseteq N$, $i \in T$, $|T| = r$, and we prove the result for $S \subseteq N$, $i \in S$, $|S| = r+1$, where $r \geq 1$.

We first prove that there exists $M > 0$ such that $D^w_i(V,S) \leq M$ for all $w \in \Delta^N_{++}$. Assume the contrary, i.e., for all $M > 0$, there exists $w \in \Delta^S_{++}$ such that

$$D^w_i(V,S) = w_i Q(w,S) > M.$$

Then, we have the following statement: for all $k = 1, 2, \cdots$, there exists $w^k \in \Delta^N_{++}$ such that

$$w^k_i Q(w^k,S) > k,$$
$$\big(Z^{w^k}(V,S) + w^k Q(w^k,S)\big) \in \partial V(S).$$

From the induction hypothesis, $Z^{w^k}(V,S)$ is bounded from below. Then, $z^k := Z^{w^k}(V,S) + w^k Q(w^k,S)$ is a sequence such that $z^k \in \partial V(S)$ for all $k = 1, 2, \cdots$, and $\lim_{k\to\infty} z^k_i = +\infty$. On the other hand, z^k is bounded from below for all $k = 1, 2, \cdots$, which contradicts N4. We can prove that there exists $m < 0$ such that $m \leq D^w_i(V,S)$ for all $w \in \Delta^N_{++}$ in a parallel manner by using N5. □

From Lemma 3, we know that, for any $V \in \mathcal{G}$, the range of the function $\xi^w(V) : \Delta^N_{++} \to \mathbb{R}^N$ is bounded. We now obtain two propositions.

Proposition 2 *Let $V \in \mathcal{G}$. Then, the function $\xi^w(V) : \Delta^N_{++} \to \mathbb{R}^N$ satisfies C3.*

Proof. Let w^k be a convergent sequence such that there exists a non-empty coalition $T \subset N$ which satisfies $\lim_{k\to\infty} w^k_j = 0$ for all $j \in T$ and $\lim_{k\to\infty} w^k_i > 0$ for all $i \in N \backslash T$.

Let $j \in T$ be fixed. Consider the sequence $D^{w^k}_j(V,S)$ for $S \subseteq N, j \in S$. From Lemma 3, there exists a convergent subsequence of $D^{w^k}_j(V,S)$ for each

$S \subseteq N, j \in S$. For notational convenience, we assume that $D_j^{w^k}(V, S)$ itself converges for all $S \subseteq N$, $j \in S$. Let a coalition $\bar{S} \subseteq N$ which satisfies $j \in \bar{S}$ and $\bar{S} \nsubseteq T$ be fixed. We prove that $\lim_{k\to\infty} D_j^{w^k}(V, \bar{S}) = 0$. From the definition, for any k,

$$D^{w^k}(V, \bar{S}) = w^k Q(w^k, \bar{S}).$$

Let $i \in \bar{S} \cap (N \backslash T)$. Suppose that $\lim_{k\to\infty} Q(w^k, \bar{S}) = +\infty$. Then, the sequence $z^k := Z^{w^k}(V, \bar{S}) + w^k Q(w^k, \bar{S})$ satisfies $z^k \in \partial V(\bar{S})$ for all k and $\lim_{k\to\infty} z_i^k = +\infty$. Since z^k is bounded from below, this result contradicts N4. Similarly, if we assume $\lim_{k\to\infty} Q(w^k, \bar{S}) = -\infty$, we can obtain the result which contradicts N5. It follows that $Q(w^k, \bar{S})$ is a bounded sequence. Since $\lim_{k\to\infty} w_j^k = 0$, we have

$$\lim_{k\to\infty} D_j^{w^k}(V, \bar{S}) = 0.$$

As a result, we obtain

$$\lim_{k\to\infty} \xi_j^{w^k}(V) = \lim_{k\to\infty} \sum_{R \subseteq T : j \in R} D_j^{w^k}(V, R) \text{ for all } j \in T.$$

It follows that $\lim_{k\to\infty} \xi_T^{w^k}(V) \in \partial V(T)$. □

Proposition 3 *Let $V \in \mathcal{G}$. Then, the function $\xi^w(V) : \Delta_{++}^N \to \mathbb{R}^N$ satisfies $C2$.*

Proof. We prove that $D^w(V, S)$ is continuous for each non-empty $S \subseteq N$. The result holds for $S = \{i\}$, $i \in N$. We proceed by induction.

Take an arbitrary sequence $\{w^k\}_{k=1}^\infty \subseteq \Delta_{++}^N$ such that $w^k \to w^* \in \Delta_{++}^N$. Let $i \in S$. From Lemma 3, there exists $M > 0$ and $m < 0$ such that for all k,

$$m \leq w_i^k Q(w^k, S) \leq M.$$

Since $w_i^k Q(w^k, S)$ is a bounded sequence and i is an arbitrary player, $w_S^k Q(w^k, S)$ is also a bounded sequence. Thus, there exists a convergent subsequence. Take an arbitrary convergent subsequence $w_S^{l(k)} Q(w^{l(k)}, S) \to w_S^* Q^*$, where $l : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. From the induction hypothesis, $Z^w(V, S)$ is continuous, which implies

$$\left(Z^{w^{l(k)}}(V, S) + w_S^{l(k)} Q(w^{l(k)}, A)\right) \to \left(Z^{w^*}(V, S) + w_S^* Q^*\right) \in \partial V(S).$$

Since $Q(w^*, S)$ is unique, we obtain $Q^* = Q(w^*, S)$, which implies $w_S^{l(k)} Q(w^{l(k)}, S) \to w_S^* Q(w^*, S)$. Since any convergent subsequence of $w_S^k Q(w^k, S)$ converges to

$w_S^* Q(w^*, S)$, we have $w_S^k Q(w^k, S) \to w_S^* P(w^*, S)$. That is, $D^{w^k}(V, S) \to D^{w^*}(V, S)$, which proves continuity of $D^w(V, S)$. □

From Propositions 2 and 3, for any $V \in \mathcal{G}$, the weighted Egalitarian solution satisfies C1 to C3. Together with Proposition 1, we obtain the following result:

Theorem 1 *For any $V \in \mathcal{G}$, $C(V) \subseteq cl\Xi(V)$.*

5 Consistent weighted Shapley value

In this section, we define a new solution concept which we call the consistent weighted Shapley value. Then, we show that the set of consistent weighted Shapley values includes the Core.

We first explain the motivation of introducing a new solution concept. In TU games, it is shown that the weighted Shapley value can be calculated as the expected value of marginal contributions. So, the result by Monderer et al. (1992) means that any element of the Core can be represented as the expected value. This result is interesting in the sense that the two concepts are defined from different perspectives. On the one hand, the Core is defined based on dominance relation, which highly relies on the result of cooperation. On the other hand, the weighted Shapley value calculates the expected value of marginal contributions, which measure individual's influence on a coalition. From the viewpoint of extending the result to NTU games, it is desirable to introduce a new solution concept which calculates the expected value of marginal contributions.

We need to extend the following two concepts: the marginal contribution and the probability distribution over the set of orders of players derived from a vector $w \in \Delta^N_{++}$. As for the probability, we can simply use the same definition as TU case. Let us review the definition.[3] Let $\mathbf{R}(N)$ denote the set of orders of players in N. For any $R \in \mathbf{R}(N)$ and $i \in N$, let $\mathbf{B}(R, i)$ denote the set of players preceding i in R. For any $R = (i_1, \cdots, i_n) \in \mathbf{R}(N)$ and $w = (w_{i_1}, \cdots, w_{i_n}) \in \Delta^N_{++}$, we define $P_w(R)$ by

$$P_w(R) = \Pi_{m=1}^n \Big(w_{i_m} / \sum_{t=1}^m w_{i_t} \Big). \tag{7}$$

If we calculate the expected value of marginal contributions by using the above probability, then we can obtain the weighted Shapley value in TU games.[4]

[3] Here, we use the same notations of Chun (1991).

[4] For detailed discussion, see Kalai and Samet (1987).

The remaining problem is to extend the marginal contribution. We deal with the problem by following Maschler and Owen (1989). In TU case, the marginal contribution represents how the attainable payoff changes from the entrance of a player. Based on this idea, we define the marginal contribution in the following way. Let an arbitrary order of players be given. The first player receives the maximum payoff which can be obtained by his own. Given the payoff of preceding players, each player receives the payoff which maximizes his payoff in the set of attainable payoffs for the coalition consisting of the player and preceding players.

We give a formal definition. For any $R \in \mathbf{R}(N)$ and $V \in \mathcal{G}$, we define $m^R(V)$ recursively as follows:

$$m_i^R(V) = \max\{x_i : x_i \in V(\{i\})\} \text{ if } \mathbf{B}(R,i) = \emptyset,$$
$$m_j^R(V) = \max\big\{x_j \in \mathbb{R} : \big(x_j, (m_i^R(V))_{i\in\mathbf{B}(R,i)}\big) \in V\big(\mathbf{B}(R,j) \cup \{j\}\big)\big\} \text{ if } \mathbf{B}(R,j) \neq \emptyset.$$

We prove a lemma which guarantees that the marginal contribution is well-defined.

Lemma 4 *Let $V \in \mathcal{G}$, $S \subseteq N$, $|S| \geq 2$, $j \in S$ and $z \in \mathbb{R}^{S\setminus\{j\}}$. Then, there exists $r \in \mathbb{R}$ such that $\big((z_k)_{k\in S\setminus\{j\}}, r\big) \in V(S)$.*

Proof. Assume not. Then, for any $r \in \mathbb{R}$, we have $\big((z_k)_{k\in S\setminus\{j\}}, r\big) \notin V(S)$. We define $Z \subseteq \mathbb{R}^S$ by

$$Z = \big\{\big((z_k)_{k\in S\setminus\{j\}}, r\big) : r \in \mathbb{R}\big\}.$$

The two sets $V(S)$ and Z are convex, and the intersection between the two sets is empty. From the separation theorem, there exists $p \in \mathbb{R}^S$, $p \neq \mathbf{0}$ such that

$$p \cdot x \leq p \cdot y \text{ for all } x \in V(S), y \in Z. \tag{8}$$

Suppose that $p_j > 0$. Then, by taking an arbitrary sequence $\{y^k\}_{k=1}^{\infty} \subseteq Z$ such that $y_j \to -\infty$, we have $p \cdot y^k \to -\infty$, which contradicts equation (8). Similarly, if we suppose that $p_j < 0$, we obtain the same contradiction. As a result, $p_j = 0$.

Suppose that $p_i < 0$ for some $i \in S\setminus\{j\}$. Consider the sequence $\{x^l\}_{l=1}^{\infty} \subseteq V(S)$ such that $x_i^l \to -\infty$ and $x_h^l = x_h^{l+1}$ for all $h \in S$, $h \neq i$, $l = 1, 2, \cdots$. From N2 (comprehensive), such a sequence always exists. Then, $p \cdot x^l \to +\infty$, which violates equation (8). It follows that $p \geq \mathbf{0}$. Since $p \neq \mathbf{0}$, there exists at least one player $i \in S\setminus\{j\}$ such that $p_i > 0$.

Let $x \in \partial V(S)$ be arbitrarily given. For any $m \in \mathbb{N}$, let $\tilde{x}^m$ denote the following vector:

$$\tilde{x}_j^m = x_j - m, \tilde{x}_i^m = x_i \text{ for all } i \in S\setminus\{j\}.$$

From N2 (comprehensive), $\tilde{x}^m \in V(S)$. Let $i \in S\backslash\{j\}$ be a player who satisfies $p_i > 0$. Then, for any $m \in \mathbb{N}$, there exists x^m such that[5]

$$\begin{aligned} &x_j^m = \tilde{x}_j^m, \\ &x_h^m = x_h \text{ for all } h \in S, h \neq i, h \neq j, \\ &x^m \in \partial V(S). \end{aligned}$$

Consider the sequence $\{x^m\}_{m=1}^{\infty} \subseteq \partial V(S)$. Since $x_j^m \to -\infty$, from N4, we have $x_i^m \to +\infty$. Since $p_j = 0$ and $p_i > 0$, we have $p \cdot x^m \to +\infty$, which contradicts equation (8). □

From Lemma 4 and N2 (closed), the marginal contribution $m^R(V)$ is well-defined for all $R \in \mathbf{R}(N)$ and $V \in \mathcal{G}$.

Given the setting above, we define a new solution concept. Let $V \in \mathcal{G}$ and $w \in \Delta_{++}^N$. We define the consistent weighted Shapley value ϕ^w as follows:

$$\phi^w(V) = \sum_{R \in \mathbf{R}(N)} P_w(R) m^R(V).$$

Let $V \in \mathcal{G}$. We define the set of consistent weighted Shapley values $\Phi(V)$ by

$$\Phi(V) = \{\phi^w(V) : w \in \Delta_{++}^N\}.$$

We prove that ϕ^w satisfies C3. Let us introduce an additional notation. For any order $R \in \mathbf{R}(N)$, $i \succ_R j$ means that i is a successor of j in the order R.

Proposition 4 *Let $V \in \mathcal{G}$. Then, the function $\phi^w(V) : \Delta_{++}^N \to \mathbb{R}^N$ satisfies C3.*

Proof. Let $\{w^k\}_{k=1}^{\infty}$ be a convergent sequence such that there exists a coalition $T \subset N$, $T \neq \emptyset$ which satisfies

$$\begin{aligned} &\lim_{k \to \infty} w_i^k = 0 \text{ for all } i \in T, \\ &\lim_{k \to \infty} w_j^k > 0 \text{ for all } j \in N\backslash T. \end{aligned}$$

For any $R \in \mathbf{R}(N)$, $\{P_{w^k}(R)\}_{k=1}^{\infty}$ is a bounded sequence, which implies that there exists a convergent subsequence. Assume, without loss of generality, that $\{P_{w^k}(R)\}_{k=1}^{\infty}$ itself converges for all $R \in \mathbf{R}$. Let $P^*(R)$ denote the limit point of $P_{w^k}(R)$ for $R \in \mathbf{R}$.

[5]The basic idea of the sequence x^m is the following: from the vector x, we decrease the payoff of j by m, while increase the payoff of i so that the resulting vector is in $\partial V(S)$. We can show that x^m always exists from N1(proper subset), N2(comprehensive) and N4.

Let $R = (i_1, \cdots, i_n) \in \mathbf{R}(N)$ be an order such that there exists a player $i \in N \backslash T$ and another $j \in T$ who satisfies

$$j \succ_R i.$$

Suppose that $j = i_{m'}$, $2 \leq m' \leq n$. From equation (7), we have

$$P^*(R) = \lim_{k \to \infty} \Pi_{m=1}^n \Big(w_{i_m}^k / \sum_{t=1}^m w_{i_t}^k \Big). \tag{9}$$

From the assumption, $\lim_{k \to \infty} w_{i_{m'}}^k = 0$ and $\lim_{k \to \infty} \sum_{t=1}^{m'} w_{i_t}^k > 0$. It follows that

$$\lim_{k \to \infty} \Big(w_{i_{m'}}^k / \sum_{t=1}^{m'} w_{i_t}^k \Big) = 0.$$

Thus, equation (9) is equal to 0. As a result, we restrict our attention to the following set of orders:

$$\mathbf{R}'(N) = \{R \in \mathbf{R}(N) : i \succ_R j \text{ for all } i \in N \backslash T \text{ and } j \in T\}.$$

We calculate the limit of the consistent weighted Shapley values.

$$\begin{aligned} \lim_{k \to \infty} \phi_T^{w^k}(V) &= \lim_{k \to \infty} \sum_{R \in \mathbf{R}(N)} P_{w^k}(R) m_T^R(V) \\ &= \sum_{R \in \mathbf{R}'(N)} P^*(R) m_T^R(V). \end{aligned}$$

For each $R \in \mathbf{R}'(N)$, we have $m_T^R(V) \in \partial V(T)$. From N2 (convex), we have $\phi_T^{w^k}(V) \in V(T)$ for each k. From N2 (closed), $\lim_{k \to \infty} \phi_T^{w^k}(V) \in V(T)$. □

Let $V \in \mathcal{G}$. Note that C2 of $\phi^w(V)$ immediately follows. From Proposition 4, $\phi^w(V)$ satisfies C3. On the other hand, $\phi^w(V)$ does not necessarily satisfy C1 for $V \in \mathcal{G}$. Since the solution function ϕ^w calculates the convex combination of marginal contributions, the resulting payoff vector might lie in the interior of $V(N)$. In order that the outcome is pareto optimal, it is sufficient to assume that $V(N)$ is the closed half-space of a hyperplane.

Together with Proposition 1, we obtain the following result:

Theorem 2 *Let $V \in \mathcal{G}$ be a game such that $V(N)$ is the closed half-space of a hyperplane. Then, $C(V) \subseteq cl\Phi(V)$.*

Theorem 2 has the following implication: any element of the Core is attainable as the expected value of marginal contributions.

6 Concluding remarks and example

In Section 5, we considered the new solution concept which calculates the expected value of marginal contribution of each player. In previous works, there is another solution concept which is defined based on the idea of the marginal contribution; the MC value by Otten et al. (1998). The basic idea of the value is to rescale the marginal contribution vector so that it belongs to the pareto frontier. We extend the value by incorporating positive weights. Let $V \in \mathcal{G}$. The MC value with positive weight w, denoted as $MC^w(V)$, is the unique payoff vector which has the following properties:

1: $MC^w(V) = \alpha\phi^w(V)$ for some $\alpha \in \mathbb{R}$.

2: $MC^w(V) \in \partial V(N)$.

$MC^w(V)$ is an extension of $\phi^w(V)$ in the sense that the value can choose pareto optimal payoff vector for any game $V \in \mathcal{G}$.

Unfortunately, we cannot express all elements of the Core by using $MC^w(V)$. We give a counter example. Consider the following game $V \in \mathcal{G}$ with player set $N = \{1, 2, 3\}$:

$$V(\{i\}) = \{x \in \mathbb{R} : x \leq 0\} \text{ for all } i \in N, V(\{1,2\}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 1\},$$
$$V(\{1,3\}) = \{(x_1, x_3) \in \mathbb{R}^2 : x_1 + x_3 \leq 0\}, V(\{2,3\}) = \{(x_2, x_3) \in \mathbb{R}^2 : x_2 + x_3 \leq 0\}.$$

Finally, $V(N)$ is the set of payoff vectors in $\mathbb{R}^3$ such that the cross-section view along each plane is represented as the following Figure 1:

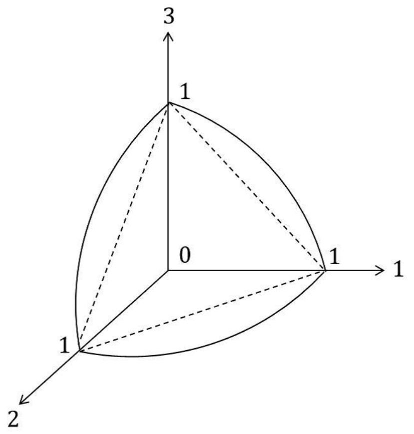

Figure 1

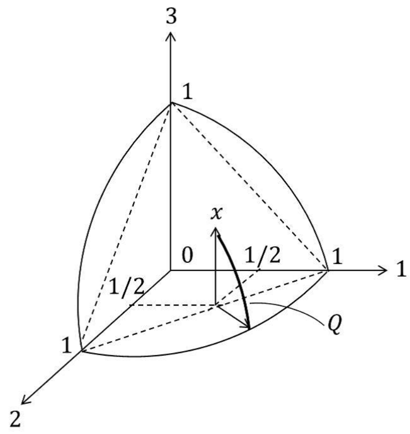

Figure 2

Let us consider the payoff vector $x = (1/2, 1/2, a)$ such that $a > 0$ and $x \in \partial V(N)$. Then, we have $x \in C(V)$. In this game, we can check that $m_3^R(V) = 0$ for all $R \in \mathbf{R}(N)$, which implies that $MC_3^w(V) = 0$ for all $w \in \Delta_{++}^N$. As a consequence, the vector x cannot be represented as the outcome of $MC^w(V)$.

On the other hand, from Theorem 1, we can express the element x by using the weighted Egalitarian solution $\xi^w(V)$. To see this, consider the weight $w^\lambda = (\frac{1}{\lambda}, \frac{1}{\lambda}, 1 - \frac{2}{\lambda})$, $\lambda \in (2, \infty)$. Then,

$$\xi_i^{w^\lambda}(V) = \frac{1}{2} + \frac{1}{\lambda} \cdot \max\Big\{t : \Big(\frac{1}{2}, \frac{1}{2}, 0\Big) + tw^\lambda \in \partial V(N)\Big\} \text{ for } i = 1, 2.$$

By letting $\lambda \to \infty$, we have $\xi_i^{w^\lambda}(V) \to \frac{1}{2}$ for $i = 1, 2$, which converges to x. By considering all possible weights w^λ, $\lambda \in (2, \infty)$, we can attain the all payoff vectors represented as the arc Q in Figure 2.

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