The Relationship between Revealed Preference and the Slutsky Matrix

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Abstract

This study provides a calculation method for utility function from a smooth demand function whose Slutsky matrix is negative semi-definite and symmetric. Moreover, this study presents an axiom of demand functions, and show that under the strong axiom, this axiom is equivalent to the existence of the corresponding continuous preference relation. If the demand function obeys this axiom, then such a preference relation is unique, and our calculating utility function represents its preference relation. These results are obtained even if the demand function is not income-Lipschitzian. Further, this study shows that the mapping from demand function into continuous preference relation is continuous, which assures the applicability of our results for econometrics. Moreover, this study shows that if this demand function satisfies the rank condition, then our utility function is smooth. Lastly, this study shows that under an additional axiom, the above results hold even if the demand function has a corner solution.

Keywords: demand function, utility function, Slutsky matrix, integrability theory, revealed preference.

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1 Introduction

In consumer theory, it is repeatedly mentioned that for any smooth demand function, the negative semi-definiteness (NSD) and the symmetry (S) of the Slutsky matrix ensures the strong axiom.[1] However, to our knowledge, there is no proof of such a 'result'.[2] We think that this 'result' is just a folklore in consumer theory.

In this paper, we close this folklore. We provide a concrete calculating method for utility function from a smooth demand functions satisfying (NSD) and (S). (Theorem 1) By using this result, we show the equivalence result between the strong axiom and assumptions of Slutsky matrix. ((NSD) and (S). Corollary 1.) Also, we show that these are equivalent to the existence of a global concave solution of the following partial differential equation:

$$Du(p) = f(p, u(p)),$$

with any initial value condition. By Shephard's lemma, this solution coincides with the **expenditure function** of corresponding preference relations. We introduce some examples for understanding our method.

In previous study such as Hurwicz and Uzawa (1971), the existence of the solution of the above equation was first proved, and then the existence of the utility function was proved. To prove such an existence theorem, previous study required an additional conditions other than both (NSD) and (S). (For example, income-Lipschitzian assumption.) In contrast, we first show the existence of the utility function, and then show the existence of the solution. Moreover, we do not need any additional requirement. Thus, we can get such an equivalence theorem.

Solving this problem, we can get a chance to obtain an application of the integrability theory to econometrics. One of the virtue of the integrability theory conparing with several related theories is the actual computability of the corresponding preference relations. Hence, there is an application on econometric theory. In general, the preference of consumer is more difficult to estimate than the demand, because there is no observed data corresponding with the preference. However, if one uses the integrability theory, he/she gets at once the estimation value of preference from the estimation value of demand. Therefore, this theory decreases such a difficulty.

However, there is a trap on this consideration. In statistics, there is an important criterion of estimation method, namely, the **consistency** of

[1]For example, see Kihlstrom, Mas-Colell, and Sonnenschein (1976), Hurwicz and Richter (1979), etc.

[2]There exist a paper that claims to verify this 'result'. However, there is no decent proof in his paper. See section 4.1 for more detailed arguments.

estimation. Suppose that x is the true value of some estimation problem, and let x_N be the estimation value with sample size N. The consistency criterion requires the estimation value for converging to the true value x **in probability** as the sample size N tends to infinity.

Suppose that there is a consistent estimation method of a demand function, and let f_N be the estimation value with sample size N. Using our theorem 1, we get an estimation value $\succsim_N$ of preference relation corresponding with f_N. However, it is unknown whether this estimation method is consistent. Is the consistency inherited?

To argue the above arguments rigorously, we should clarify the **topology** on the spaces of demand functions and preference relations, because the definition of the consistency includes a topological notion, namely, the convergence. We want to use the local C^1 topology on the space of demand functions, and the closed convergence topology on the space of preference relations. To set these topologies, we become to be able to argue the consistency of above estimation method.

However, here is another problem. The closed convergence topology is only defined on the set of **continuous** preference relations. Is our estimation value $\succsim_N$ of preference continuous? Before answering this problematic question, we should mention that it is not sure that there exists a corresponding continuous preference relation with given demand function, even if this demand function satisfies the strong axiom.[3]

Therefore, we need to add an axiom on revealed preference relation, named the **NLL axiom**. We show that under the strong axiom, the NLL axiom is equivalent to the existence of corresponding continuous preference relation. (Theorem 2) We will argue later that the interpretation of the NLL axiom is very natural: it only rules out the lexicographic-like behavior on (direct) revealed preference relation. Moreover, in this case such a preference is unique, and the calculated utility function by our method is continuous and represents it. (Theorem 3)

Comparing the results in related literatures, our result is one of the sharpest results. Actually, to our knowledge there is no "necessary and sufficient" result for ensuring the continuity of the calculated preference. See

[3]Here, we shall survey such results in several research that treats the derivation to the preference from given demand function. At first, Uzawa (1960) only showed the upper semi-continuity of the indirect revealed preference relation. Hurwicz and Uzawa (1971) also showed the upper semi-continuity of their preference relation, and presents three sufficient condition of the lower semi-continuity, though it is not necessary. Richter (1966) only treat the existence theorem. Debreu (1972) and Hosoya (2013) provides sufficient conditions for the smoothness of the derived preference, which is clearly not necessary for continuity.

section 2.4 for more detailed arguments.

Hence, there is a mapping combining the demand function satisfying both the strong and the NLL axioms with the corresponding unique continuous preference relation. We show the continuity of this mapping, (theorem 4) and thus we can answer the above question affirmatively. If there is a consistent estimation method for demand function, then the corresponding estimation method for preference relation is also consistent. (Corollary 2)

Adding to these results, we can show the smoothness of our utility function. (Theorem 5) Actually, if there exists a smooth and regular utility function corresponding with given demand function, then our utility function is also smooth and regular. Hosoya (2013) showed that if a demand function satisfies the rank condition together with (NSD) and (S), then there exists a corresponding smooth and regular utility function. Therefore, our method also constructs a smooth utility function if the demand function satisfies the rank condition. (Corollary 3)

All results above are obtained under assumptions of the non-emptiness and the smoothness for demand function. However, there is an important class of the demand functions we cannot treat, that is, the demand function corresponding with the **quasi-linear utility**. In this case, there must be a **corner solution**, and it make the problem more difficult. We consider such a case, and derive the generalized result of theorems 1-4 under the additional axiom, named the **C axiom**. (Theorems 6-8 and corollary 4) That is, we present the following results: for any smooth demand functions, (NSD), (S) and the C axiom holds if and only if there is a corresponding continuous preference relation. If so, then such a preference relation is unique, and f satisfies the NLL axiom. Moreover, the mapping from the demand function into the corresponding continuous preference relation is continuous with respect to certain topologies. Note that the C axiom is not so strong, and in the previous setup, it is equivalent to the NLL axiom. See theorem 2 for more detailed arguments.

Theorem 1 and corollary 1 are obtained when consumption space is any subset of $\mathbb{R}^n_+$. In contrast, the rest results are obtained when the consumption space is $\mathbb{R}^n_{++}$, and f is surjective. Although these constraints are so strong, there is a reason for assuming these. In section 4.2, we explain that if these constraints are not satisfied, how strange results are obtained.

In section 2.1, we introduce definitions of several words, which include 'demand function', 'weak and strong axiom', 'Slutsky matrix', 'utility function', and especially, 'preference relation corresponding with given demand function'. In section 2.2, we give the formal statements of theorem 1, which provides the concrete calculating method for utility function from demand function. Also, in this section, corollary 1 is obtained, which states the equivalence between the requirements of Slutsky matrix and the strong axiom of revealed preference. In section 2.3, we introduce some examples for actual calculation. In section 2.4, we argue the existence of the corresponding continuous preference relation, the uniqueness of it, and the continuity of the mapping from the demand function into the preference relation. Also, the smoothness of our utility function is argued. In section 3, we generalize such results for demand functions with non-full domain under the C axiom. In section 4.1, we discuss the relationship between this study and past researches in this context. In section 4.2, we explain why our assumptions are needed by showing several examples. Section 5 is the conclusion. The proofs of all results are in section 6.

2 Main Results

2.1 Preliminary

We consider that the notation Ω denotes the consumption space, and assume that Ω is a subset of $\mathbb{R}^n_+$, where $n \ge 2$. Although many study assumes that Ω is either $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n | x_i > 0, i = 1, ..., n\}$ or $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \ge 0, i = 1, ..., n\}$,[4] we do not need this assumption for a while.

Choose any binary relation $\succsim$ on Ω, that is, $\succsim \subset \Omega^2$. We write $x \succsim y$ if $(x, y) \in \succsim$ and $x \not\succsim y$ if $(x, y) \notin \succsim$. We say that $\succsim$ is

- **complete** if for any $x, y \in \Omega$, either $x \succsim y$ or $y \succsim x$,
- **transitive** if for any $x, y, z \in \Omega$, $x \succsim y$ and $y \succsim z$ imply $x \succsim z$,
- **continuous** if $\succsim$ is closed in Ω^2,
- **strongly monotone** if for any $x, y \in \Omega$, $x \succsim y$ and $y \not\succsim x$ when $x \gneq y$.

We call a binary relation $\succsim$ on Ω a **preference relation** if it is complete and transitive. If $\succsim$ is a preference relation, then we write $x \succ y$ if $x \succsim y$ and $y \not\succsim x$, and $x \sim y$ if $x \succsim y$ and $y \succsim x$.

Suppose that $u : \Omega \to \mathbb{R}$ satisfies the following condition:

$$u(x) \ge u(y) \Leftrightarrow x \succsim y.$$

[4] Throughout this paper, the subscript notation x_i means the i-th coordinate of the vector x, and the superscript notation f^i means the i-th coordinate of the function f.

Then, we say that u represents $\succsim$, or u is a **utility function** of $\succsim$. Note that if some function u represents $\succsim$, then $\succsim$ is a preference relation, and $\succsim$ is continuous if u is continuous.[5]

Next, we call a function $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \Omega$ a *demand function* if it satisfies the homogeneity of degree zero and Walras' law: that is,

$$f(ap, am) = f(p, m), \forall a > 0,$$
$$p \cdot f(p, m) = m,$$

for any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$.[6]

Suppose that f is a demand function. Then, the following relations can be defined.

$$\begin{aligned} x \succ_r y &\Leftrightarrow x \neq y, \exists (p, m), x = f(p, m) \text{ and } p \cdot y \le m, \\ x \succ_{ir} y &\Leftrightarrow \exists x_0, ..., x_k \in \Omega, x_0 = x, x_k = y, \\ &\quad\ \text{and } x_{i+1} \succ_r x_i \text{ for any } i = 0, ..., k-1. \end{aligned}$$

Then, f satisfies the *weak axiom* if $\succ_r$ is asymmetric (that is, $x \succ_r y$ implies $y \not\succ_r x$), and f satisfies the *strong axiom* if $\succ_{ir}$ is asymmetric. Clearly, the strong axiom implies the weak axiom.

Now, let $\succsim$ be a binary relation on Ω and define

$$f^{\succsim}(p, m) = \{x \in \Omega | \forall y, p \cdot y \le m \Rightarrow x \succsim y\}.$$

Then, $f^{\succsim}$ is homogeneous of degree zero, and if $\succsim$ is strongly monotone, then $f^{\succsim}$ satisfies Walras' law. We call $f^{\succsim}$ a *demand relation induced by* $\succsim$ and say that $\succsim$ corresponds with f (or, f corresponds with $\succsim$) if $f = f^{\succsim}$. If u represents $\succsim$, then $f^{\succsim}$ is sometimes written as f^u, and we say that u corresponds with f (or, f corresponds with u) if $f^u = f$. It is well known that for any demand function f, $f = f^{\succsim}$ for some preference relation $\succsim$ if and only if f satisfies the strong axiom.[7]

Suppose that f is a C^1-class demand function. Let

$$s_{ij}(p, m) = \frac{\partial f^i}{\partial p^j}(p, m) + \frac{\partial f^i}{\partial m}(p, m) f^j(p, m),$$

and define $(n \times n)$-matrix $S_f(p, m) = (s_{ij}(p, m))_{i,j=1}^n$. This matrix is called the **Slutsky matrix** of f. We say that f satisfies (NSD) if $S_f(p, m)$ is negative semi-definite for any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, and (S) if $S_f(p, m)$ is symmetric for any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$.

[5] Conversely, if a preference relation $\succsim$ is continuous, then there is a continuous function u that represents $\succsim$. This result is obtained by the second countability of Ω. See Debreu (1954).

[6] Actually, our theorem 1 does not need the homogeneity of f. The homogeneity is automatically satisfied in the setup of theorem 1.

[7] See Richter (1966) or Mas-Colell, Whinston, and Green (1995).

2.2 Main Results

Theorem 1. Suppose that f is a C^1-class demand function that satisfies (NSD) and (S). Fix any $\bar{p} \in \mathbb{R}^n_{++}$, and choose any $x \in \Omega$. If there is no (p, m) such that $x = f(p, m)$, then define $u_{f,\bar{p}}(x) = 0$. Otherwise, choose any (p, m) such that $f(p, m) = x$, and consider the following differential equation:

$$\dot{c} = f((1-t)p + t\bar{p}, c) \cdot (\bar{p} - p), \tag{1}$$

with the initial condition $c(0) = m$. Then, there exists a solution $c : [0, 1] \to \mathbb{R}_{++}$ of above equation, and $c(1)$ is independent to the choice of (p, m). Define

$$u_{f,\bar{p}}(x) = c(1).$$

Then, $f = f^{u_{f,\bar{p}}}$.

As a corollary, we can obtain the following result.

Corollary 1. Suppose that f is a C^1-class demand function. Then, the following two statements are equivalent.

(I) f satisfies (NSD) and (S).

(II) f satisfies the strong axiom.

(III) For any $(p^*, m^*) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, there exists a **concave** solution $u : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ of the following differential equation

$$Du(p) = f(p, u(p)), \tag{2}$$

with initial value condition $u(p^*) = m^*$.

Remarks on Theorem 1 and Corollary 1. In the proof, we will show at first the existence of the utility function, (theorem 1) and then the existence of the solution of (2). (corollary 1) In contrast, in past researches the existence of the solution of (2) was first proved, and then the existence of the utility function was verified.[8] Actually, two approaches are not so different. Readers will see in our lemma 9 that lemma 2 is actually equivalent to the existence of the solution of (2), where lemma 2 is needed for proving theorem 1.

Note that, to prove lemma 2, it is needed not only (S), but also lemma 1, and lemma 1 needs (NSD). Therefore, our existence theorem of the solution of (2) requires both (NSD) and (S). In contrast, previous researches in

[8]For example, see Katzner (1970) or Hurwicz and Uzawa (1971).

this context showed (2) by (S) and some additional conditions like income-Lipschitzian, and use (NSD) to prove that $f = f^{u_{f,\bar{p}}}$. Therefore, our contribution is to show by using (NSD), we can remove additional conditions to show the existence of the solution of (2).

In fact, the function $u_{f,\bar{p}}$ has an interesting interpretation. Choose an income-consumption curve $d : m \mapsto f(\bar{p}, m)$. Then, for any $x \in \Omega$, there uniquely exists m^* such that $f(\bar{p}, m^*)$ is indifferent to x, and $u_{f,\bar{p}}(x) = m^*$. This interpretation arises from the proof of corollary 1: define an expenditure function

$$E(p) = \inf\{p \cdot y | u_{f,\bar{p}}(y) \geq u_{f,\bar{p}}(x)\}.$$

By Shephard's lemma, (lemma 7) we have E satisfies (2), and thus $t \mapsto E((1-t)p + t\bar{p})$ satisfies (1) for any (p, m) with $x = f(p, m)$. Therefore, $u_{f,\bar{p}}(x) = E(\bar{p})$. By definition, we have $u_{f,\bar{p}}(f(\bar{p}, m)) = m$ for any $m > 0$, and especially $u_{f,\bar{p}}(f(\bar{p}, m)) = u_{f,\bar{p}}(x)$ if and only if $m = u_{f,\bar{p}}(x)$, which implies that the above interpretation is correct.

2.3 Examples of Calculation

Example 1(Cobb-Douglas case). Let $\alpha_i \in]0, 1[$ for $i = 1, ..., n$ and $\sum_i \alpha_i = 1$, and consider the following demand function:

$$f^i(p, m) = \frac{\alpha_i m}{p_i}.$$

Let $\bar{p} = (1, 1, ..., 1)$. Then, the differential equation in theorem 1 is

$$\dot{c}(t) = \sum_i \frac{\alpha_i(1-p_i)}{p_i + t(1-p_i)} c(t), c(0) = m.$$

Therefore, we have

$$\begin{aligned} c(1) &= c(0)e^{\int_0^1 \sum_i \frac{\alpha_i(1-p_i)}{p_i+t(1-p_i)}dt} \\ &= c(0)e^{-\sum_i \alpha_i \log p_i} = m\prod_i p_i^{-\alpha_i}. \end{aligned}$$

Choose any $x \in \mathbb{R}^n_{++}$. If we set $m = 1$, then we can easily verify that $x = f(p, m)$ if and only if $p_i = \frac{\alpha_i}{x_i}$. Therefore,

$$u_{f,\bar{p}}(x) = C \times x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n},$$

where $C > 0$ is some constant.

Example 2(CES case). Let $\alpha_i \in]0,1[$ for $i = 1, ..., n$ and $\sum_i \alpha_i = 1$, and $\rho \in]-\infty, 1[\setminus\{0\}$. Consider the following demand function:

$$f^i(p,m) = \frac{\alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{-1}{1-\rho}} m}{\sum_j \alpha_j^{\frac{1}{1-\rho}} p_j^{\frac{-\rho}{1-\rho}}}.$$

Let $\bar{p} = (1, 1, ..., 1)$. Then, the differential equation in theorem 1 is

$$\dot{c}(t) = \frac{\sum_i \alpha_i^{\frac{1}{1-\rho}} (p_i + t(1-p_i))^{\frac{-1}{1-\rho}} (1-p_i)}{\sum_i \alpha_i^{\frac{1}{1-\rho}} (p_i + t(1-p_i))^{\frac{-\rho}{1-\rho}}} c(t), c(0) = m.$$

Therefore, we have

$$\begin{aligned} c(1) &= c(0) e^{\int_0^1 \frac{\sum_i \alpha_i^{\frac{1}{1-\rho}} (p_i + t(1-p_i))^{\frac{-1}{1-\rho}} (1-p_i)}{\sum_i \alpha_i^{\frac{1}{1-\rho}} (p_i + t(1-p_i))^{\frac{-\rho}{1-\rho}}} dt} \\ &= c(0) e^{\frac{1-\rho}{\rho} (\log \sum_i \alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{-\rho}{1-\rho}} - \log \sum_i \alpha_i^{\frac{1}{1-\rho}})} \\ &= mC[\sum_i \alpha_i^{\frac{1}{1-\rho}} p_i^{\frac{-\rho}{1-\rho}}]^{\frac{1}{\rho}-1}, \end{aligned}$$

where $C > 0$ is some constant. Choose any $x \in \mathbb{R}^n_{++}$. If we set $\frac{m}{\sum_j \alpha_j^{\frac{1}{1-\rho}} p_j^{\frac{-\rho}{1-\rho}}} = 1$, then we can easily verify that $x = f(p, m)$ if and only if $p_i = \alpha_i x_i^{\rho-1}$, and in such a case, we have $m = \sum_i \alpha_i x_i^\rho$. Therefore,

$$u_{f,\bar{p}}(x) = C \times [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho + ... + \alpha_n x_n^\rho]^{\frac{1}{\rho}}.$$

2.4 Application: Transmission of the Consistency

Hereafter, we assume that $\Omega = \mathbb{R}^n_{++}$.

Consider an estimation problem of some true value x, where x is in some topological space. We assume that an estimation method of x is already obtained. Then, the estimation value x_N of x with data size N is a measurable function from some probability space into the space of x. This estimation method is said to be **consistent** if x_N converges to x in probability: that is, for any neighborhood U of x, the probability of the event $\{x_N \notin U\}$ converges to 0 as $N \to \infty$.

Now, we consider the local C^1 topology in the topology of the space of demand functions, and the closed convergence topology in the topology of the space of (continuous) preference relations. Then, we can argue that the consistency of estimation methods in demand functions and (continuous) preference relations.

Consider an estimation method of f, and suppose that every estimation value f_N satisfies (NSD) and (S). Then, by theorem 1, we can calculate a preference relation $\succsim_N$ such that $f_N = f^{\succsim_N}$. Therefore, we obtain the estimation value $\succsim_N$ of the preference relation. Our question is here: *is the consistency inherited?* When we have already obtained the consistent estimation method of f, can we get the consistent estimation method of $\succsim$ in the above manner?

Clearly, above problem is nonsense if $\succsim_N$ is not continuous, because the closed convergence topology cannot be defined. However, the strong axiom is not sufficient for the continuity of $\succsim_N$. Moreover, the uniqueness of the $\succsim_N$ is also needed: If $\succsim_N$ is not unique, we cannot determine what preference is better for the estimation value.

The uniqueness of continuous preference relation $\succsim$ such that $f = f^{\succsim}$ is called the **recoverability** of f.[9] Later we will show that the recoverability easily fails if f is not **surjective**. Hence, hereafter we assume that all demand functions we treat are surjective.

Now, suppose that f is a surjective demand function. We introduce an additional axiom in f. We say that f satisfies the **NLL axiom**[10] if for every $x \in \Omega$, $i, j \in \{1, ..., n\}$ with $i \neq j$, there exists $y \in \Omega$ such that,

i) $y_k = x_k$ if $i \neq k \neq j$,

ii) $y_i < x_i$ and $y_j > x_j$, and

iii) $y \succ_r x$.

This axiom rules out the possibility the commodity i is so special that if $y_i < x_i$, then x is preferred than y even if y_j is extremely high. This is the reason of the name "NLL".

Define $G(x) = \{p \in \mathbb{R}^n_{++} | \sum_{i=1}^n p_i = 1, f(p, p \cdot x) = x\}$. We call G the **inverse demand correspondence** of f. The following two theorems hold.

Theorem 2. Suppose that f is a C^1-class surjective demand function. Then, the following four statements are equivalent.

(i) $f = f^{\succsim}$ for a continuous preference relation $\succsim$.

(ii) f satisfies (NSD) and (S), and $u_{f,\bar{p}}$, defined in Theorem 1 is continuous, strongly increasing and strictly quasi-concave.

[9] See Mas-Colell (1977).

[10] "NLL" is the abbreviation of "Non-lexicographiclike".

(iii) f satisfies both the strong and the NLL axioms.

(iv) f satisfies the strong axiom, and G is compact- and convex-valued, and u.h.c..[11]

Theorem 3. Suppose that f is a C^1-class surjective demand function such that $f = f^{\succsim}$ for some continuous preference relation $\succsim$. Then, $\succsim$ is represented by $u_{f,\bar{p}}$ and f is recoverable.

Now, let $\mathscr{F}$ denote the set of all C^1-class surjective demand function satisfying the strong and the NLL axiom. Then, for any $f \in \mathscr{F}$, there exists a unique continuous preference relation $H(f)$ such that $f = f^{H(f)}$. Note that by theorem 1 and 2, $u_{f,\bar{p}}$ represents $H(f)$.

Theorem 4. H is continuous: that is, for any sequence (f_n) of $\mathscr{F}$ such that $f_n \to f$ with respect to local C^1 topology, $u_{f_n,\bar{p}}$ converges to $u_{f,\bar{p}}$ with respect to local uniform topology, and $H(f_n) \to H(f)$ with respect to the closed convergence topology.

Corollary 2. If f_N converges to f in probability, then $H(f_N)$ converges to $H(f)$ in probability.[12]

Hence, the answer of our question is "yes, under the NLL axiom".

At last, we can show the following result.

Theorem 5. Suppose that $k \geq 1$, f is a C^k-class demand function such that $f = f^u$ for some C^k-class regular (that is, $Du(x) \neq 0$ for any x) function $u : \Omega \to \mathbb{R}$. Then, $u_{f,\bar{p}}$ is also C^k-class and regular.

Meanwhile, theorem 2 of Hosoya (2013) showed the following result: suppose that $f \in \mathscr{F}$ and f is C^k-class. If f satisfies the rank condition, (that is, the rank of $S_f(p,m)$ is always $n-1$,) then there is a C^k-class regular function u such that $f = f^u$. Therefore, we obtain the following corollary.

Corollary 3. If $f \in \mathscr{F}$ is C^k-class and satisfies the rank condition, then $u_{f,\bar{p}}$ is C^k-class and regular.

Remarks on Theorems and Corollaries. In general, to obtain the continuity of corresponding preference relation is difficult in both revealed preference theory and integrability theory. For example, Uzawa (1960) only

[11] Later, we name this property of G the C axiom.

[12] Obviously, "in probability" can be replaced with "almost everywhere".

showed the upper semi-continuity of the preference relation defined by $x \succsim v \Leftrightarrow v \not\succ_{ir} x$ under income-Lipschitzian setups. Hurwicz and Uzawa (1971) also showed only the upper semi-continuity of our $u_{f,\bar{p}}$, and for the lower semi-continuity, they only presented three sufficient conditions, which include the usual boundary condition of demand, the single-valuedness of the inverse demand correspondence, and somewhat odd condition on boundary structure.

In contrast, we present a necessary and sufficient condition which ensures the existence of the continuous preference relation corresponds with f, namely, the NLL axiom. Under this axiom, our $u_{f,\bar{p}}$ itself represents the unique continuous preference relation corresponding with f. The interpretation of the NLL axiom is very clear. Suppose that $f = f^{\succsim}$ for some continuous $\succsim$. Let e_i be the i-th unit vector. Because f is surjective, we have $x+e_j \succ_r x$, and thus $x + e_j \succ x$. By continuity of $\succsim$, we have $x + e_j - \varepsilon e_i \succ x$ for some $\varepsilon > 0$, and thus, $\succsim$ is not 'lexicographiclike'. The NLL axiom represents this property in the language of the revealed preference theory.

Note that the fourth statement of theorem 2 is an extension of the second sufficient condition of Hurwicz-Uzawa: in fact, if G is single-valued, then we can show that it is continuous, and thus fourth requirement of theorem 2 holds. Comparing with the NLL axiom, this statement is more practical: in many situation, the fourth statement is much easier to confirm than the NLL axiom.

We claimed in theorem 3 the recoverability of the smooth demand function. Our theorem does not assume the income-Lipschitzian property. In contrast, Mas-Colell (1977) showed in his theorem 3 that for any $f = f^u$ with continuous, monotone, and strictly quasi-concave utility function u, f is recoverable if f is income-Lipschitzian. We sought an example of $f \in \mathscr{F}$ such that f is not income-Lipschitzian, but we could not find it. Therefore, it is vague whether our result is independent to Mas-Colell's one. However, at least our result is not obtained at once by Mas-Colell's one, and probably this result is not known.

Later we will show in theorem 8 that in wider class than $\mathscr{F}$, the continuity of H still holds. However, in its class we cannot define $u_{f,\bar{p}}$, and thus it is impossible to argue the convergence of $u_{f_k,\bar{p}}$. Therefore, theorem 4 is independent to theorem 8.

Hurwicz and Uzawa (1971) showed an example of smooth demand function corresponding with a utility function with **kinked** indifference curve. For such utility function, the inverse demand correspondence must be multi-valued. In contrast, Hosoya (2013) showed that if f satisfies the rank condition and the weak axiom, then the inverse demand correspondence must be single-valued. Therefore, this example does not satisfies the rank condition.

This shows that our theorems are independent to theorem 2 of Hosoya (2013) and theorem 3 of Hosoya (2015).

Readers may think that the assumption $\Omega = \mathbb{R}^n_{++}$ or the surjectivity of f is odd. However, if this assumption is dropped, there are many problematic phenomena. See section 4.2 for more detailed arguments.

3 Extension: Demand Function with Non-Full Domain

Consider a utility function $u(x_1) + x_2$, where u is a C^1-class increasing function, and u' is decreasing. The corresponding demand function is $f^1(p, m) = (u')^{-1}(p_1/p_2), f^2(p, m) = \frac{1}{p_2}[m - p_1 f^1(p, m)]$. However, if $m > 0$ is not sufficiently large, then $m - p_1 f^1(p, m) \le 0$. Because we assume that $\Omega = \mathbb{R}^n_{++}$, in this case we must consider that $f(p, m)$ is undefined, and thus our results in previous section cannot be applied.

One may think that this problem arises from the fact $\Omega = \mathbb{R}^n_{++}$. However, the problem remains even if $\Omega = \mathbb{R}^n_+$. In this case, we have $f(p, m) = (\frac{m}{p_1}, 0)$ if $m \le p_1(u')^{-1}(p_1/p_2)$, and thus this demand function is not smooth at any (p, m) with $m = p_1(u')^{-1}(p_1/p_2)$. Therefore, again our results in previous section cannot be applied.

Hence, we should extend the notion of the demand function. In this section, $f : A \to \Omega$ is called a demand function if it satisfies the homogeneity of degree zero and Walras' law, and the domain A is an open cone of $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$.

We now introduce a new axiom, called the **C axiom**.[13] A surjective demand function f satisfies the C axiom if the inverse demand correspondence $G(x) = \{p | \sum_i p_i = 1, x = f(p, p \cdot x)\}$ is convex- and compact-valued, and u.h.c..

Note that by theorem 2, if $A = \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, then the C axiom is equivalent to the NLL axioms under the strong axiom. Moreover, Hosoya (2013) showed that if f satisfies the rank condition and (NSD), then G is a single-valued smooth function, and the C axiom holds.

Theorem 6. Let f be a demand function. Then, the following two statements are equivalent.

(I) f satisfies (NSD), (S), and the C axiom.

(II) There exists a continuous preference relation $\succsim$ such that $f = f^{\succsim}$.

[13] C is the abbreviation of "convex, compact, and continuous".

Moreover, if (I) or (II) is satisfied, then f satisfies the NLL axiom, and such continuous $\succsim$ is unique.

Now, we will introduce a topology on the space $\mathscr{F}'$ of all surjective C^1-class demand functions satisfying (NSD), (S), and the C axiom, called the **local C^1 topology**. Choose any $f \in \mathscr{F}'$, and let A be the domain of f. Define $U(f,i,\varepsilon)$ be the set of all $f' \in \mathscr{F}'$ satisfying the following conditions:

(i) the domain of f' includes C_i, where

$$C_i = \{(p,m) \in \mathbb{R}^n_{++}\times\mathbb{R}_{++} | \|(p,m)\| \in [\frac{1}{i}, i], \inf_{(q,w)\notin A} \|(q,w)-(p,m)\| \ge \frac{1}{i}\}.$$

(ii) $\|f' - f\|_{C^1} < \varepsilon$, where $\|\cdot\|_{C^1}$ is the C^1 norm on C_i.

The local C^1 topology is the least topology such that all $U(f,i,\varepsilon)$ are open.

Theorem 7. The local C^1 topology is well-defined, Hausdorff, and first countable topology on $\mathscr{F}'$. A sequence (f_k) on $\mathscr{F}'$ converges to $f \in \mathscr{F}'$ with respect to this topology if and only if for any compact C included in the domain of f, it is included in the domain of f_k for sufficiently large k, and $\|f_k - f\|_{C^1}$ converges to 0 on C as $k \to \infty$.

Choose $f \in \mathscr{F}'$. By theorem 6, there uniquely exists a continuous preference relation $H(f)$ such that $f = f^{H(f)}$.

Theorem 8. The function H is continuous: that is, if (f_k) converge to f with respect to the local C^1 topology, then $H(f_k) \to H(f)$ with respect to the closed convergence topology.[14]

Corollary 4. If f_k converges to f in probability, then $H(f_k)$ converges to $H(f)$ in probability.

Remarks on Theorems and Corollary. Theorem 6 corresponds with theorems 1, 2 and 3, and Theorem 8 corresponds with theorem 4. Because $u_{f,\bar{p}}$ cannot be defined, we cannot obtain the corresponding result with theorem 5. However, in the proof of theorem 6, a utility function u_v appears. If f satisfies the rank condition, then this u_v is smooth because it has the same definition as u^g_v in Hosoya (2013).

We should mention that the definition of the local C^1 topology. By theorem 7, the restriction of this topology on $\mathscr{F}$ defined the previous section

[14]Note that by theorem 7, the local C^1 topology is first countable, and thus the continuity follows from this sequential requirement.

is the usual local C^1 topology. Therefore, theorem 8 rigorously corresponds with theorem 4.

4 Discussion

4.1 Relationship to Related Literatures

The most near research in the related literatures are Richter (1979). He stated in his theorem 12 that if a demand function f is C^1-class and satisfies (NSD) and (S), then there exists a utility function u such that $f = f^u$. However, he did not provide a rigorous proof, and only provide the 'sketch'. In his 'sketch', the existence of the solution of (2) (corollary 1 of ours) was mentioned. However, he only said that it can be shown by the 'dual' arguments in Debreu (1972).

We cannot understand the meaning of this 'dual' arguments. However, we think that his rough 'sketch' is primordially wrong. The reason is the following: to prove the existence of the solution of (2), (or lemma 2, these are equivalent,) we should use (NSD) for proving lemma 1. However, Debreu (1972) used only the Jacobi's integrability condition of the inverse demand function. Samuelson (1950) showed that this condition corresponds with not (NSD), but (S) only. Therefore, we think that by Richter's idea, corollary 1, and hence his theorem, cannot be proved.[15]

Next, Hurwicz and Uzawa (1971) showed in their theorem 2 that if a demand function f is differentiable, and satisfies (NSD), (S), and **income-Lipschitzian** requirement, then there exists a utility function u such that $f = f^u$. However, their EXISTENCE THEOREM I (states the existence of the solution of (2) on any compact set) used in the proof of theorem 2 is doubtful. They used the formula

$$\frac{\partial}{\partial y}\int_0^1 g(x,y)dx = \int_0^1 \frac{\partial g}{\partial y}(x,y)dx,$$

for some g, while g is only differentiable. If g is continuously differentiable, then the above formula is well-known and called the Leibniz's integral rule. However, any extension of this rule requires some additional condition on $\frac{\partial g}{\partial y}$, and we think that Hurwicz-Uzawa's setup may deviate such a requirement. Meanwhile, if f is C^1-class, then their result shrinks the Nikliborc's theorem, (Nikliborc (1929)) and their theorem 2 holds. However, if f is assumed to be

[15] Also, we doubt Tsuji's extension theorem used in his paper.

C^1-class, then our corollary 1 is an extension of their result because in our setup the income-Lipschitzian properties is not required.[16]

Thirdly, Mas-Colell (1977) showed that if $f = f^u$ for some continuous, monotone and strictly quasi-concave utility function u, then f is recoverable if f is income-Lipschitzian. We discussed in section 2 the relationship between this result and our theorem 3.

Theorem 2 of Hosoya (2013) showed that if $f \in \mathscr{F}'$ is C^1-class, and satisfies the rank condition, (NSD) and (S), then there exists a C^1-class utility function u such that $f = f^u$. This result required the rank condition because the inverse demand correspondence must be single-valued and C^1-class. It is easy to show that if inverse demand correspondence is single-valued and C^1-class, then the rank condition holds. In contrast, our theorem 6 admit the multi-valuedness of the inverse demand correspondence, and thus the rank condition is not needed. In section 2, we argued that there exists $f \in \mathscr{F}$ that does not satisfy the rank condition. Therefore, our theorem 6 is independent to this result.

Hurwicz and Richter (1979) showed if f is C^1-class, then the condition (S) is characterized by some axiom in revealed preference. Kihlstrom, Mas-Colell, and Sonnenschein (1976) showed that if f is C^1-class, then the weak axiom is stronger than (NSD). These two researches said that if f satisfies (NSD) and (S), then the strong axiom holds. However, the proof was not provided in both papers.

Finally, Hosoya (2013) showed that if $f : A \to \Omega$ is C^1-class and satisfies the rank condition and the weak axiom, then there uniquely exists a complete, **p-transitive**, and continuous binary relation $\succsim^f$ on Ω such that $f = f^{\succsim^f}$. Hosoya (2015) showed that the mapping $f \mapsto \succsim^f$ is continuous. This result is a variety of our theorem 8.

4.2 On the Consumption Space

One may think that the assumption $\Omega = \mathbb{R}^n_{++}$ is odd, and $\Omega = \mathbb{R}^n_+$ is more natural. However, to set $\Omega = \mathbb{R}^n_+$, we cannot assume that f is surjective, because if so, f should have a non-smooth point. The following two examples show that the lack of surjectivity causes very problematic results.

Example 3. Let $n = 2$ and $\succsim_1$ be a lexicographic order; that is, $x \succsim_1 y$ if and only if either $x_1 > y_1$ or $x_1 = y_1, x_2 \geq y_2$ holds. Let $\succsim_2$ be represented by $u(x) = x_1$, and $\succsim_3$ by $u(x) = x_1 - x_2$. Then, all preference relations lead

[16] We note that their theorem 1 is also doubtful: if f is not continuously differentiable but only differentiable, then the solution u of (2) may not be C^2-class, and thus $S_f(p, u(p)) = D^2u(p)$ may not be symmetric.

the same demand function

$$f(p,m)=(\frac{m}{p_1},0).$$

By theorem 1, we can calculate

$$u_{f,\bar{p}}(x)=\begin{cases}x_1 & \text{if } x_2=0,\\ 0 & \text{otherwise,}\end{cases}$$

if $\bar{p}_1=1$. But this function represents none of $\succsim_i$. Especially, theorem 1 can calculate the false utility function though the true order is lexicographic.[17] Also, clearly $\succsim_2 \neq \succsim_3$ though both preferences are continuous. This example shows that the recoverability property fails if the range of demand function is too small.

Example 4. Consider the following function:

$$h_1(c)=\begin{cases}e^{-\frac{1}{c^2}} & \text{(if } c>0,)\\ 0 & \text{(otherwise.)}\end{cases}$$

It is well-known that this function is C^∞-class, increasing on $[0,+\infty[$, and $h_1(c)\to 1$ if $c\to\infty$. Next, let

$$h_2(c)=1-\frac{h_1(1-c)}{h_1(1)},$$

$$h_3(c)=\tan(\frac{\pi}{2}h_1(c-2)),$$

$$h(c)=h_2(c)+h_3(c).$$

Then, h is a C^∞-class nondecreasing function, $h(c)=1$ if $c\in[1,2]$, increasing on $[0,+\infty[\setminus[1,2]$, $h(c)\downarrow 0$ if $c\downarrow 0$, and $h(c)\uparrow\infty$ if $c\uparrow\infty$.

Next, let $x_1,x_2>0$ and consider the following equation:

$$(x_1^{\frac{1}{1+\frac{1}{c}}}+x_2^{\frac{1}{1+\frac{1}{c}}})^{1+\frac{1}{c}}=h(c).$$

Because the derivative of the left-hand side of this equation with respect to c is negative, we have the solution $c(x_1,x_2)$ of this equation is unique. By the implicit function theorem, we have $c(x_1,x_2)$ is C^∞-class. As the left-hand

[17] It is known that $\succsim_1$ does not have any utility function. See Kreps (1988).

side is increasing in both x_1 and x_2, we have $c(x_1, x_2)$ is increasing in both x_1 and x_2. Let

$$(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow c(x_1, x_2) \geq c(y_1, y_2).$$

Then, the indifference curve of $\succsim$ is

$$L(c) = \{(x_1, x_2) \in \mathbb{R}^2_{++} | x_1^{\frac{1}{1+\frac{1}{c}}} + x_2^{\frac{1}{1+\frac{1}{c}}} = h(c)^{\frac{1}{1+\frac{1}{c}}}\},$$

and thus we can verify that it is strictly convex toward the origin and has non-zero Gaussian curvature.[18] For any $p_1, p_2, m > 0$, define $(x_1(t), x_2(t)) = (t, \frac{m - p_1 t}{p_2})$. Then $\frac{d}{dt} c(x_1(t), x_2(t))$ is positive if t is sufficiently small, and is negative if $\frac{m}{p_1} - t$ is sufficiently small. Therefore, we have $f^{\succsim}$ is a single-valued C^{∞}-class mapping from $\mathbb{R}^2_{++} \times \mathbb{R}_{++}$ onto $\mathbb{R}^2_{++}$.

However, for any $c \in [1, 2]$, $(0, 1)$ and $(1, 0)$ are the limit points of $L(c)$. This indicates that there is no continuous and transitive extension of $\succsim$ on $\mathbb{R}^2_+$. In fact, if $\succsim^*$ is such an extension, we have

$$(\frac{1}{4}, \frac{1}{4}) \sim^* (0, 1) \sim^* (\frac{1}{2^{3/2}}, \frac{1}{2^{3/2}}),$$

but $c(\frac{1}{4}, \frac{1}{4}) < c(\frac{1}{2^{3/2}}, \frac{1}{2^{3/2}})$, a contradiction.[19]

Figure 1 illustrates the above example. Note that if the consumption space Ω is $\mathbb{R}^2_{++}$, then $f^{\succsim}$ satisfies all assumptions in Theorem 2 and 3, and thus the corresponding continuous preference relation is unique. This example show that even if the range of f is dense in $\mathbb{R}^n_+$, there may be no continuous preference relation $\succsim^*$ on $\mathbb{R}^n_+$ such that $f = f^{\succsim^*}$.

5 Conclusion

We presented a concrete calculating method for utility function from a smooth demand function satisfying (NSD) and (S). Moreover, we presented an axiom, named the NLL axiom, which is equivalent to the existence of continuous preference relation corresponding with this demand function. If so, such a preference relation is unique, and our calculating utility function represents it. Further, the mapping from demand function to this unique continuous preference relation is continuous. At last, if the demand function satisfies some additional requirement, then our utility function is smooth.

[18] This condition assures the differentiability of the demand function. See Debreu (1972).

[19] This $\succsim$ has a unique **upper semi-continuous** and transitive extension. However, we cannot state whether this property is general or not.

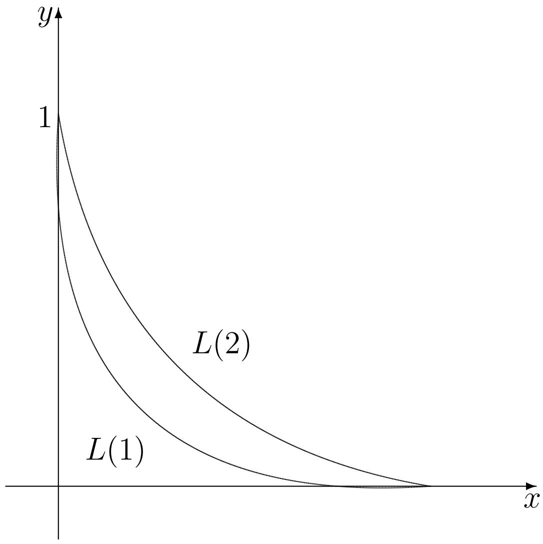

Figure 1: Illustration of example.

These results were obtained if the domain of the demand function is full. However, there exists demand functions with non-full domain such as that corresponding with some quasi-linear preference relation. We presented an axiom, named the C axiom, and showed that if demand function satisfies this axiom, then all the above results can be obtained even if the domain of this function is not full.

There are a few future tasks. At first, we may remove the smoothness assumption of demand functions. In equation (1), we get the uniqueness of the equation is obtaind if f is **locally Lipschitz** with respect to m. Therefore, we think that the C^1 assumption of f is too strong. However, if f is not C^1-class, then (NSD) and (S) cannot be defined. Hence, we need an alternative assumptions.

Secondly, we consider the demand function with the strong axiom. What happens under only the weak axiom? In other words, what happens if (S) is not satisfied? Hosoya (2013) showed in this setup there uniquely exists a complete, **p-transitive**, and continuous binary relation corresponding with given demand function. However, this result requires the rank condition. What happens if the rank condition is dropped? This is an open problem.

Thirdly, there may be a wider class of recoverable demand functions than both C^1 and income-Lipschitzian class. We expect any demand function with **locally** income-Lipschitzian property is recoverable.

6 Proofs

6.1 Preliminary: the Knowledge of the Differential Equation

Hereafter, we frequently use the knowledge in the theory of ordinary differential equations. This includes the following facts: consider the following ordinary differential equation

$$\dot{x} = f(t, x), x(t_0) = x_0,$$

where f is defined on an open set $\Gamma \subset \mathbb{R}^2$, $(t_0, x_0) \in \Gamma$, and both f and $\frac{\partial f}{\partial x}$ are continuous. Under these conditions, there exists a C^1-class function $x(t)$ defined on an open interval including t_0 such that $\dot{x}(t) = f(t, x(t))$ and $x(t_0) = x_0$. We call such a function x a **solution** of this equation with initial value condition $x(t_0) = x_0$. If x_1, x_2 are two solution with the same initial value condition, then $x_1(t) = x_2(t)$ whenever t is included in the intersection of the domains of those functions. For two solutions x_1, x_2 with the same initial value condition, x_2 is called the **extension** of x_1 if the domain of x_2 includes that of x_1. A solution x is **nonextendable** if there is no extension of x except x itself. If x is nonextendable and $]a, b[$ is the domain of x, then for any compact subset $C \subset \Gamma$, there exists $t_*, t^* \in]a, b[$ such that $(t, x(t)) \notin C$ whenever either $a < t < t_*$ or $t^* < t < b$.

Next, consider the following ordinary differential equation

$$\dot{x} = f(t, x; y), x(t_0) = x_0,$$

where f is also defined on an open set $\Gamma \subset \mathbb{R}^2 \times \mathbb{R}^k$, $(t_0, x_0, y) \in \Gamma$, and both f and $\frac{\partial f}{\partial x}$ are continuous. Under these conditions, there exists a solution $x(t)$ such that $x(t_0) = x_0$. Suppose that $x(t; y, t_0, x_0)$ is the nonextendable solution for fixed (y, t_0, x_0), and the domain of x in t is $]a, b[$. For any $c, d \in]a, b[$ with $c < t_0 < d$, there exists an open neighborhood U of (y, t_0, x_0) such that if $(z, t_1, x_1) \in U$, then there exists a solution $x(t; z, t_1, x_1)$ defined on $[c, d]$ such that $x(t_1; z, t_1, z_1) = x_1$. Moreover, the function $(t, y, t_0, x_0) \to x(t; y, t_0, x_0)$ is continuous. If all $\frac{\partial f}{\partial y_i}, i = 1, ..., k$ are continuous, then this function is continuously differentiable on (t, y, t_0, x_0), and

$$\frac{\partial^2 x}{\partial t \partial y_i} = \frac{\partial^2 x}{\partial y_i \partial t}.$$

These facts are in many standard textbook on ordinary differential equations.[20]

[20] For example, see Pontryagin (1965), Hartman (1997), or Smale and Hirsch (1974).

6.2 Proof of Theorem 1

At first, we should modify the equation (1). Consider the following equation:

$$\dot{c} = f((1-t)p + tq, c) \cdot (q - p), c(0) = m, \tag{3}$$

where $p, q \in \mathbb{R}^n_{++}$ and $m > 0$. The next lemma is needed.

Lemma 1. Suppose that $w : [0, \bar{t}] \to \mathbb{R}_{++}$ is the solution of (3). Let $p(t) = (1-t)p + tq, x = f(p, m)$ and $y = f(p(\bar{t}), w(\bar{t}))$. Then, $p \cdot y \geq m$ and $p(\bar{t}) \cdot x \geq w(\bar{t})$.

Proof. Let $c(t) = p \cdot f(p(t), w(t))$. Then, by simple calculation we have

$$\dot{c}(t) = p^T S_f(p(t), w(t))(q - p),$$

where the superscript T means the transpose. Meanwhile, by Walras' law,

$$p(t)^T S_f(p(t), w(t))(q - p) = 0.$$

To subtract the latter from the former, we have

$$\dot{c}(t) = -t(q - p)^T S_f(p(t), w(t))(q - p) \geq 0,$$

by (NSD). Therefore, $p \cdot y = c(\bar{t}) \geq c(0) = m$ and the first inequality holds. The proof of the second inequality is symmetrical to the first and we omit it. ■

Lemma 2. There exists a solution c of (3) defined on $[0, 1]$.

Proof. Suppose not. Let $t^* > 0$ be the supremum of t such that the solution c of (1) can be defined on $[0, t]$. Then, the solution $c : [0, t^*[\to \mathbb{R}_{++}$ with $c(0) = m$ can be defined. Define $p(t) = (1-t)p + tq$. Because c cannot be extended, for any compact set $C \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, there exists $\bar{t} \in [0, t^*[$ such that $(p(t), c(t)) \notin C$ for any $t \in [\bar{t}, t^*[$. However, $p(t) \in [p, q] \subset \mathbb{R}^n_{++}$ if $t \in [0, 1]$, and thus we have either $\limsup_{t \uparrow t^*} c(t) = \infty$ or $\liminf_{t \uparrow t^*} c(t) = 0$.

Define $x = f(p, m)$. By lemma 1, we have $p(t) \cdot x \geq c(t)$ for any $t \in [0, t^*[$, and thus $\limsup_{t \uparrow t^*} c(t) < \infty$, which implies that there exists a sequence (t_k) such that $t_k \uparrow t^*$ and $c(t_k) \to 0$ as $k \to \infty$. Let $x_k = f(p(t_k), c(t_k))$. By lemma 1, $p(t_k) \cdot x \geq c(t_k) = p(t_k) \cdot x_k$ and $p \cdot x_k \geq m = p \cdot x$. This implies that $q \cdot x \geq q \cdot x_k$, and thus the sequence (x_k) is bounded. Taking subsequence, we can assume that $x_k \to x^* \in \mathbb{R}^n_+$. Because $p \cdot x^* \geq m > 0$, we have $x^* \neq 0$. Then,

$$c(t_k) = p(t_k) \cdot x_k \to p(t^*) \cdot x^* > 0,$$

a contradiction. ■

We call the solution $c : [0,1] \to \mathbb{R}_{++}$ of equation (3) the **connecting solution** with (p,q,m).

Lemma 3. Suppose c is the connecting solution of (3) with (p,q,m), and d is that of (q,p,w). Then, for any $t \in [0,1]$, $c(t) \geq d(1-t)$ if and only if $c(1) \geq w$.

Proof. Define $c_1(t) = c(1-t)$. Then, we can easily show that c_1 is the connecting solution of (3) with $(q,p,c(1))$. Suppose that $c(1) \geq w$. If $c(t) < d(1-t)$, then $c_1(1-t) < d(1-t)$. Because $c_1(0) \geq d(0) = w$, by intermediate value theorem there exists $s \in [0,t]$ such that $c_1(s) = d(s)$. By the uniqueness of the solution of the differential equation, we have $c_1 \equiv d$, and thus $c_1(1-t) = d(1-t)$, a contradiction. By the same arguments, we can show that if $c(1) < d(0) = w$, then $c(t) < d(1-t)$ for any $t \in [0,1]$. ■

Lemma 4. Suppose that $x \neq y, x = f(p,m), y = f(q,w)$ and c is the connecting solution of (3) with (p,q,m). If $w \geq c(1)$, then $p \cdot y > m$.

Proof. Suppose at first that $w > c(1)$. Let d be the connecting solution of (3) with (q,p,w). Then, we have $d(0) > c(1)$, and by lemma 3, $d(1) > c(0)$. By lemma 1, $p \cdot y \geq d(1)$, and thus $p \cdot y > m$.

Next, suppose that $w = c(1)$. As in the proof of lemma 1, let $p(t) = (1-t)p + tq$ and $d(t) = p \cdot f(p(t), c(t))$. Then,

$$\dot{d}(t) = -t(q-p)^T S_f(p(t), c(t))(q-p) \geq 0.$$

Therefore, $p \cdot y = d(1) \geq d(0) = m$. If $p \cdot y = m$, then $\dot{d} \equiv 0$ on $[0,1]$. Let $S_t = S_f(p(t), c(t))$ and $c_1^t, ..., c_n^t$ be eigenvalues of S_t. By (NSD), we have $c_i^t \leq 0$, and by (S), there exists an orthogonal matrix P^t such that

$$P_t^T S_t P_t = \begin{pmatrix} c_1 & 0 & ... & 0 \\ 0 & c_2 & ... & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & ... & c_n \end{pmatrix}.$$

Let $d_i = \sqrt{-c_i}$ and

$$A_t = P_t \begin{pmatrix} d_1 & 0 & ... & 0 \\ 0 & d_2 & ... & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & ... & d_n \end{pmatrix} P_t^T.$$

Then, $-S_t = A_t^2$ and A_t is symmetric. Then, $0 = -(q-p)^T S_t(q-p) = \|A_t(q-p)\|^2$, and thus $A_t(q-p) = 0$. Hence, we have $S_t(q-p) = 0$. Now, define $x(t) = f(p(t), c(p(t)))$. Then,

$$\dot{x}(t) = S_t(q-p) = 0,$$

and thus we have $y = x(1) = x(0) = x$, which contradicts our initial assumption. ■

Lemma 5. f satisfies the weak axiom.

Proof. Suppose that $x \neq y, x = f(p,m), y = f(q,w)$ and $p \cdot y \leq m$. Let c be the connecting solution of (3) with (p,q,m), and d be that with (q,p,w). By the contraposition of lemma 4, we have $c(1) > w = d(0)$, and thus by lemma 3, $m = c(0) > d(1)$. Therefore, again by lemma 4, we have $q \cdot x > w$. ■

Lemma 6. Suppose that c_1 is the connecting solution of (3) with $(p, \bar{p}, m)$, c_2 is that with (p,q,m) and c_3 is that with $(q, \bar{p}, c_2(1))$. Then, $c_1(1) = c_3(1)$.

Proof. Let $d_s(t)$ be the connecting solution of (3) with $(\bar{p}, p(s), c_1(1))$. Note that $d_0(t) = c_1(1-t)$. Let $w(t) = d_t(1)$. We will show that $w(t)$ is the connecting solution of (3) with (p,q,m). At first, $w(0) = d_0(1) = c_1(0) = m$. Next, consider the following equation:

$$\dot{x} = f((1-t)\bar{p} + tr, x) \cdot (r - \bar{p}), x(0) = m.$$

By lemma 2, there exists a solution $x(t;r)$ defined on $[0,1] \times \mathbb{R}^n_{++}$. Moreover, $d_s(t) = x(t; p(s))$ and $w(t) = x(1, p(t))$. By general arguments on ordinary differential equation, x is continuously differentiable and $\frac{\partial^2 x}{\partial r_j \partial t} = \frac{\partial^2 x}{\partial t \partial r_j}$.

Define $h^j(t,s) = \frac{\partial x}{\partial r_j}(t;r) - tf^j((1-t)\bar{p}+tr, x(t;r))$. Then, by (S),[21]

$$\begin{aligned}
\dot{h}^j(t,r) &= \frac{\partial}{\partial r_j}(f\cdot(r-\bar{p})) - f^j - t\sum_i\left[\frac{\partial f^j}{\partial p_i}+\frac{\partial f^j}{\partial m}f^i\right](r_i-\bar{p}_i)\\
&= f^j + \sum_i (r_i-\bar{p}_i)\left[t\frac{\partial f^i}{\partial p_j}+\frac{\partial f^i}{\partial m}\frac{\partial x}{\partial r_j}\right] - f^j - t\sum_i(r_i-\bar{p}_i)\left[\frac{\partial f^j}{\partial p_i}+\frac{\partial f^j}{\partial m}f^i\right]\\
&= t\sum_i(r_i-\bar{p}_i)\left[\frac{\partial f^i}{\partial p_j}-\frac{\partial f^j}{\partial p_i}-\frac{\partial f^j}{\partial m}f^i\right]+\sum_i\frac{\partial f^i}{\partial m}\frac{\partial x}{\partial r_j}(r_i-\bar{p}_i)\\
&= \left[\frac{\partial x}{\partial r_j}-tf^j\right]\sum_i(r_i-\bar{p}_i)\frac{\partial f^i}{\partial m}\\
&= h^j(t,r)\sum_i(r_i-\bar{p}_i)\frac{\partial f^i}{\partial m},
\end{aligned}$$

and thus, $\dot{h}^j(t,r) = a(t,r)h^j(t,r)$ for some continuous function $a(t,q)$. Therefore, we have

$$h(t,r) = h(0,r)e^{\int_0^t a(s,r)ds} = 0,$$

because $h(0,r)=0$. Hence, we have $\frac{\partial x}{\partial r_j}(t;r) = tf^j((1-t)\bar{p}+tr, x(t;r))$, and thus

$$\dot{w}(t) = \sum_i \frac{\partial x}{\partial r_i}(1,p(t))(q_i-p_i) = f(p(t),w(t))\cdot(q-p),$$

which implies that w is the connecting solution of (3) with (p,q,m).

Therefore, we have $w(1)=c_2(1)$. This implies that $c_3(t)=d_1(1-t)$, and thus $c_3(1)=d_1(0)=c_1(1)$, which completes the proof. ■

Now, we complete the preparation of the proof of theorem 1. Suppose that $x=f(p,m)=f(q,w)$. Define $p(t)=(1-t)p+tq$ and $c(t)=(1-t)m+tw = p(t)\cdot x$. We will show that $f(p(t),c(t))=x$ for any $t\in[0,1]$. Suppose not. Then, there exists $t\in]0,1[$ such that $f(p(t),c(t))=y\neq x$. By Walras' law, we have $p(t)\cdot y = c(t) = p(t)\cdot x$. By lemma 5, $p\cdot y>m$ and $q\cdot y>w$, and thus $p(t)\cdot y>c(t)$, a contradiction. Therefore,

$$\dot{c}(t) = x\cdot(q-p) = f(p(t),c(t))\cdot(q-p), c(0)=m,$$

and thus, c is the connecting solution of (3) with (p,q,m). Let c_1 be the connecting solution of (3) with $(p,\bar{p},m)$ and c_2 be that with $(q,\bar{q},w)$. Then, by lemma 6, we have $c_1(1)=c_2(1)$. Therefore, the definition of $u_{f,\bar{p}}(x)$ is independent to the choice of (p,m).

[21]Here, we abbreviate $f((1-t)\bar{p}+tr, x(t;r))$ as f.

Next, suppose that $x = f(p, m), p \cdot y \leq m$ and $x \neq y$. If y is not included in the image of f, then $u_{f,\bar{p}}(y) = 0 < u_{f,\bar{p}}(x)$. If $y = f(q, w)$ for some (q, w), let c be the connecting solution of (3) with (p, q, m). Then, by contraposition of lemma 4, we have $c(1) > w$. Let c_1 be the connecting solution of (3) with $(q, \bar{p}, c(1))$ and c_2 be that with $(q, \bar{p}, w)$. Applying lemma 3 for $c_1(1-t)$ and $c_2(t)$, we have $c_1(1) > c_2(1) = u_{f,\bar{p}}(y)$, and by lemma 6, $c_1(1) = u_{f,\bar{p}}(x)$. Therefore, $x = f^{u_{f,\bar{p}}}(p, m)$, which completes the proof of theorem 1. ■

6.3 Proof of Corollary 1

If f satisfies (NSD) and (S), then $f = f^{u_{f,\bar{p}}}$, and thus the strong axiom holds.

Next, if f satisfies the strong axiom, then there exists a preference relation $\succsim$ such that $f = f^{\succsim}$. For any (p^*, m^*), let $x = f(p^*, m^*)$ and define

$$E(p) = \inf\{p \cdot y | y \succsim x\}.$$

Then, the following lemma holds.

Lemma 7. $E(p)$ is concave, $E(p^*) = m^*$ and $DE(p) = f(p, E(p))$.[22]

Proof. Fix any $\varepsilon > 0$, choose any $p_1, p_2 \in \mathbb{R}^n_{++}, t \in [0, 1]$ and suppose $y \succsim x$ and $p \cdot y \leq E(p) + \varepsilon$, where $p = (1-t)p_1 + tp_2$. Then,

$$E(p) + \varepsilon \geq p \cdot y = (1-t)p_1 \cdot y + tp_2 \cdot y \geq (1-t)E(p_1) + tE(p_2).$$

Because $\varepsilon > 0$ is arbitrary, we have E is concave, and thus continuous.

Next, suppose that $y \succsim x$ and $y \neq x$. Because $x = f(p^*, m^*) = f^{\succsim}(p^*, m^*)$, we have $p \cdot y > m^*$. Meanwhile, $x \succsim x$ and $p^* \cdot x = m^*$. This implies that $E(p^*) = m^*$.

Define $x(p) = f(p, E(p))$. This function is continuous and $p \cdot x(p) = E(p)$. Fix any $\varepsilon > 0$ and define $x_\varepsilon(p) = f(p, E(p) + \varepsilon)$. By definition of $E(p)$, there exists $y \in \Omega$ such that $y \succsim x$ and $p \cdot y < E(p) + \varepsilon$. This implies that $x_\varepsilon(p) \succsim y$, and therefore $x_\varepsilon(p) \succsim x$. Hence, for any $p, q \in \mathbb{R}^n_{++}$, we have $p \cdot x(p) = E(p) \leq p \cdot x_\varepsilon(q)$. If $\varepsilon \downarrow 0$, then $x_\varepsilon(q) \to x(q)$ and thus $p \cdot x(p) \leq p \cdot x(q)$.

Now, let e_i be i-th unit vector and $p(t) = p + te_i$. Then,

$$\begin{aligned} E(p(t)) - E(p) &= (p + te_i) \cdot x(p + te_i) - p \cdot x(p) \\ &= p \cdot (x(p + te_i) - x(p)) + tx^i(p + te_i) \\ &\geq tf^i(p + te_i, E(p + te_i)). \end{aligned}$$

[22] The last statement is a variety of Shephard's lemma.

Therefore,

$$\lim_{t\downarrow 0}\frac{E(p(t))-E(p)}{t}\ge f^i(p,E(p))\ge \lim_{t\uparrow 0}\frac{E(p(t))-E(p)}{t},$$

where both limits exist and $\lim_{t\downarrow 0}\frac{E(p(t))-E(p)}{t}\le \lim_{t\uparrow 0}\frac{E(p(t))-E(p)}{t}$ because E is concave. This means that $\frac{\partial E}{\partial p_i}(p)=f^i(p,E(p))$, and thus we have $DE(p)=f(p,E(p))$, as desired. ■

Therefore, $E:\mathbb{R}^n_{++}\to\mathbb{R}_{++}$ is a concave solution of the partial differential equation (2) with $E(p^*)=m^*$.

Lastly, suppose that for any (p^*,m^*), there exists a concave function u such that

$$Du(p)=f(p,u(p)),u(p^*)=m^*.$$

Then, u is C^2-class concave function, and

$$D^2u(p^*)=S_f(p^*,m^*).$$

Therefore, $S_f(p^*,m^*)$ is negative semi-definite and symmetric. Since (p^*,m^*) is arbitrary, we have f satisfies (NSD) and (S). This completes the proof. ■

6.4 Proof of Theorem 2

At first, we shall confirm that the strong axiom implies the convexity of the set $G(x)$. Suppose that f satisfies the strong axiom. Then $f=f^{\succsim}$ for some preference relation $\succsim$. Choose any $x\in\Omega$ and $p_1,p_2\in G(x)$, and let $p\in[p_1,p_2]$. Then, $x\neq y, p\cdot y\le p\cdot x$ implies $p_i\cdot y\le p_i\cdot x$ for either $i=1$ or $i=2$, and thus $x\succ_r y$. Therefore, we have $x\succ y$, and thus $f(p,p\cdot x)=f^{\succsim}(p,p\cdot x)=x$ and $p\in G(x)$. Thus, $G(x)$ is convex.

It is clear that (ii) implies (i).

Suppose that (i) holds and $f=f^{\succsim}$, where $\succsim$ is continuous. Then, the strong axiom holds and $G(x)$ is convex for any $x\in\Omega$. Choose any $x\in\Omega$, and let e_i denote i-th unit vector. Then, by surjectivity, $x+e_j\succ_r x$, and thus $x+e_j\succ x$. Therefore, by continuity of $\succsim$, for any sufficiently small $\varepsilon>0$, we have $x+e_j-\varepsilon e_i\succ x$ for any $i,j\in\{1,...,n\}$ with $i\neq j$. This implies that $x\not\succ_r x+e_j-\varepsilon e_i$, and thus $\frac{p_j}{p_i}\ge\varepsilon$ for any $p\in G(x)$. Clearly $G(x)$ is closed by continuity of f, and thus this implies that $G(x)$ is compact. Next, choose any x, open U including $G(x)$, and a sequence (x^k) with $x^k\to x$. Suppose that there exists (p^k) such that $p^k\in G(x^k)$ and $p^k\notin U$ for infinitely many k. Taking subsequence, we can assume that $p^k\to p^*\in\mathbb{R}^n_+\setminus U$ and $p^k\notin U$ for any k. Because $\sum_i p^k_i=1$, we have there exists i with $p^*_i>0$. Because

$x + e_j - \varepsilon e_i \succ x$, we have $x + e_j - \varepsilon e_i \succ x_k$ for any sufficiently large k. This implies that for any sufficiently large k, $\frac{p_j^k}{p_i^k} \ge \frac{\varepsilon}{2}$, and thus $\frac{p_j^*}{p_i^*} \ge \frac{\varepsilon}{2}$. This implies that $p^* \in \mathbb{R}^n_{++}$. Now, $f(p^*, p^* \cdot x) = \lim_{k\to\infty} f(p^k, p^k \cdot x^k) = \lim_{k\to\infty} x^k = x$, and thus $p^* \in G(x) \subset U$, a contradiction. Therefore, we have G is u.h.c., and (iv) holds.

Now, suppose (iv) holds. Choose any $x \in \Omega$, a sequence (t_k) with $t_k \downarrow 0$, and $p^k \in G(x + e_j - t_k e_j)$. Taking subsequence, we can assume that $p^k \to p^*$. Because G is u.h.c., we have $p^* \in G(x + e_j)$. Then, $p^* \cdot (x + e_j) > p^* \cdot x$, and thus $p^k \cdot (x + e_j - t_k e_i) > p^k \cdot x$ for sufficiently large k. Therefore, if we define $y = x + e_j - t_k e_i$ for such k, we have $y \succ_r x$ and the NLL axiom holds. Thus, (iii) is correct.

Conversely, suppose (iii) holds. Because of the strong axiom, we have $G(x)$ is convex for any $x \in \Omega$. By the NLL axiom, for any $x \in \Omega$ and $i, j \in \{1, ..., n\}$ with $i \neq j$, there exists $y \in \Omega$ such that $y_i < x_i, y_j > x_j$ and $y_k = x_k$ if $i \neq k \neq j$, and $y \succ_r x$. By the strong axiom, we have $x \not\succ_r y$. Therefore, there exists $\varepsilon > 0$ such that $\frac{p_j}{p_i} \ge \varepsilon$ for any $p \in G(x)$. This implies that $G(x)$ is compact. Now, choose any $x \in \Omega$, open U including $G(x)$, and a sequence (x^k) in Ω with $x^k \to x$. Suppose that there exists (p^k) such that $p^k \in G(x^k)$ and $p^k \notin U$ for infinitely many k. Taking subsequence, we can assume that $p^k \to p^* \in \mathbb{R}^k_+ \setminus U$ and $p^k \notin U$ for any k. By the NLL axiom, for any $i, j \in \{1, ..., n\}$ with $i \neq j$ and $p_i^* \neq 0$, we have $y \succ_r x + e_j$ for some y such that $y_i < x_i, y_j > x_j + 1$ and $y_\ell = x_\ell$ if $i \neq \ell \neq j$. Then, there exists $q \in G(y)$ such that $q \cdot y \ge q \cdot x + q_j$. Then, for any sufficiently large k, we have $q \cdot y \ge q \cdot x_k + \frac{q_j}{2}$. By the strong axiom, we have

$$p^k \cdot x^k < p^k \cdot y = p^k \cdot x + (y_j - x_j) p_j^k - (x_i - y_i) p_i^k.$$

Therefore,

$$\frac{p_j^k}{p_i^k} > \frac{x_i - y_i}{y_j - x_j} + \frac{p^k \cdot x^k - p^k \cdot x}{(y_j - x_j) p_i^k},$$

where the right-hand side converges to $2\varepsilon \equiv \frac{x_i - y_i}{y_j - x_j} > 0$ as $k \to \infty$. This implies that $\frac{p_j^k}{p_i^k} \ge \varepsilon$ if k is sufficiently large, and thus $\frac{p_j^*}{p_i^*} \ge \varepsilon$. Therefore, $p^* \in \mathbb{R}^n_{++}$. By continuity of f, we have $f(p^*, p^* \cdot x) = x$, and thus $p^* \in G(x) \subset U$, a contradiction. Hence, we have G is u.h.c., and (iv) holds.

Therefore, we have (iii) is equivalent to (iv). Now, suppose (iii) and (iv) holds. Fix any $\bar{p} \in \mathbb{R}^n_{++}$ and remember the differential equation (1):

$$\dot{c} = f((1-t)p + t\bar{p}, c) \cdot (\bar{p} - p), c(0) = m.$$

Let $c(t; p, m)$ be the solution of the above differential equation with parameter (p, m). We have $u_{f,\bar{p}}(x) = c(1; p, m)$ if $x = f(p, m)$. By the general arguments of ordinary differential equation, c is continuous in (p, m). Now, suppose that there exists a sequence (x^k) in Ω such that $x^k \to x$ and $u_{f,\bar{p}}(x^k) \not\to u_{f,\bar{p}}(x)$. Choose any $p^k \in G(x^k)$. Taking subsequence, we can assume that $p^k \to p^* \in G(x)$. Then,

$$u_{f,\bar{p}}(x^k) = c(1; p^k, p^k \cdot x^k) \to c(1; p^*, p^* \cdot x) = u_{f,\bar{p}}(x),$$

a contradiction. Therefore, $u_{f,\bar{p}}$ is continuous. Next, choose any $x, y \in \Omega$ with $x \lneq y$, and $p \in G(y)$. Then $p \cdot x < p \cdot y$ and thus $u_{f,\bar{p}}(x) < u_{f,\bar{p}}(y)$, and thus $u_{f,\bar{p}}$ is strongly monotone. Lastly, let $x, y \in \Omega$, and $z \in [x, y]$ with $x \neq z \neq y$. Choose any $p \in G(z)$. Then, we have either $p \cdot x \le p \cdot z$ or $p \cdot y \le p \cdot z$. In the former case, we have $u_{f,\bar{p}}(z) > u_{f,\bar{p}}(x)$. In the latter, we have $u_{f,\bar{p}}(z) > u_{f,\bar{p}}(y)$. Therefore, $u_{f,\bar{p}}$ is strictly quasi-concave, and thus (ii) holds. This completes the proof of theorem 2. ■

6.5 Proof of Theorem 3

Let $f = f^{\succsim}$ for some complete, transitive, and continuous binary relation $\succsim$. If $x \gneq y$, then $x \succ_r y$ by surjectiveness, and thus $x \succ y$. Hence, $\succsim$ is strongly monotone.

Define

$$E^x(p) = \inf\{p \cdot y | y \succsim x\}.$$

By lemma 7, we have E^x is concave and satisfies

$$DE^x(p) = f(p, E^x(p)).$$

Choose any (p, m) with $x = f(p, m)$. Again by lemma 7, we have $E^x(p) = m$. Fix any $\bar{p} \in \mathbb{R}^n_{++}$ and define $c(t) = E^x((1-t)p + t\bar{p})$. Then, $c(t)$ satisfies the following differential equation;

$$\dot{c} = f((1-t)p + t\bar{p}, c) \cdot (\bar{p} - p), c(0) = m.$$

Therefore, we have $E^x(\bar{p}) = c(1) = u_{f,\bar{p}}(x)$.

Next, put $y = f(\bar{p}, E^x(\bar{p}))$. We will show that $x \sim y$. If $y \succ x$, then by continuity of $\succsim$, there exists $z \ll y$ such that $z \succsim x$. Meanwhile, we have $\bar{p} \cdot y = E^x(\bar{p}) > \bar{p} \cdot z$, which contradicts the definition of E^x. Therefore, we have $x \succsim y$. To show the converse, choose $y(\varepsilon) = f(\bar{p}, E^x(\bar{p}) + \varepsilon)$ for any $\varepsilon > 0$. By definition of $E^x(\bar{p})$, there exists $z \in \Omega$ such that $z \succsim x$ and $\bar{p} \cdot z \le E^x(\bar{p}) + \varepsilon$. Therefore, $y(\varepsilon) \succsim z \succsim x$. Because $\lim_{\varepsilon \downarrow 0} y(\varepsilon) = y$, we have by continuity of $\succsim$ that $y \succsim x$.

Now, choose any $x, y \in \Omega$ and define $z = f(\bar{p}, E^x(\bar{p})), w = f(\bar{p}, E^y(\bar{p}))$. By above arguments, we have $z \sim x$ and $w \sim y$. If $x \succsim y$, then $z \succsim w$, and thus

$$u_{f,\bar{p}}(x) = E^x(\bar{p}) = \bar{p} \cdot z \geq \bar{p} \cdot w = E^y(\bar{p}) = u_{f,\bar{p}}(y).$$

Conversely, suppose $u_{f,\bar{p}}(x) \geq u_{f,\bar{p}}(y)$. Then, $\bar{p} \cdot z \geq \bar{p} \cdot w$ and thus either $z = w$ or $z \succ_r w$. Hence, we have $z \succsim w$ and thus $x \succsim y$. Therefore, $u_{f,\bar{p}}$ represents $\succsim$ and thus such a $\succsim$ is unique. ■

6.6 Proof of Theorem 4

Suppose that (f_k) is a sequence of $\mathscr{F}$ converges to $f \in \mathscr{F}$ with respect to local C^1 topology: that is, for any compact set $C \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++}$,

$$\sup_{(p,m)\in C} \|f_k(p,m) - f(p,m)\| + \sup_{(p,m)\in C} \|Df_k(p,m) - Df(p,m)\| \to 0.$$

Let $D \subset \Omega$ be a compact set, and choose any $x \in D$. We will show that there exists an open neighborhood U of x and $\varepsilon > 0$ such that if $f_k(p, p \cdot x) \in U$, then $p_i \geq \varepsilon$. Suppose not. Then, there exists $p^\ell \in \mathbb{R}^n_{++}$ and $z^\ell = f_{k(\ell)}(p^\ell, p^\ell \cdot x)$ such that $\sum_j p^\ell_j = 1, z^\ell \to x$ and $p^\ell_i \to 0$. Suppose at first that $k(\ell) = k$ for infinitely many ℓ. Taking subsequence, we can assume that $k(\ell) = k$ for any ℓ. By theorem 2, we have the inverse demand correspondence G_k of f_k is compact-valued and u.h.c.. Then, $p^\ell \in G_k(z^\ell)$ and $z^\ell \to x$, and thus $p^\ell_i \geq \varepsilon$ for some $\varepsilon > 0$, a contradiction. Hence, we can assume that $k(\ell)$ is increasing. We can assume that $p^\ell \to p^* \in \mathbb{R}^n_+$, where $\sum_j p^*_j = 1$. Choose any j with $p^*_j > 0$.

By the NLL axiom of f, there exists $y \in \Omega$ such that

1) $y_j < x_j, y_i > x_i + 1$, and $y_l = x_l$ if $i \neq l \neq j$.

2) $y = f(q, q \cdot y)$ and $q \cdot (x + e_i) \leq q \cdot y$ for some $q \in \mathbb{R}^n_{++}$, where e_i is i-th unit vector.

Now, define $y^k = f_k(q, q \cdot y)$. Then, $y^k \to y$. By assumption, $q \cdot z^\ell + \frac{q_i}{2} \leq q \cdot y$ for sufficiently large ℓ, and thus we have $q \cdot z^\ell \leq q \cdot y^{k(\ell)}$ for sufficiently large ℓ. However,

$$\lim_{\ell\to\infty} p^\ell \cdot z^\ell = p^* \cdot x > p^* \cdot y = \lim_{\ell\to\infty} p^\ell \cdot y^{k(\ell)},$$

and thus we can show that $p^\ell \cdot z^\ell \geq p^\ell \cdot y^{k(\ell)}$ if ℓ is sufficiently large, which contradicts the strong axiom of $f_{k(\ell)}$. Therefore, our claim is correct. Define U_x be such a neighborhood. Then (U_x) is an open covering of D, and thus we can show that there exists $\varepsilon > 0$ such that if $f_k(p, p \cdot x) = x$ for some $x \in D$, then $p_i \geq \varepsilon$ for all i.

Let $C = \{p | \sum_{i=1}^n p_i = 1, f(p, p\cdot x) = x \text{ for some } x \in D\}$ and $C_k = \{p | \sum_{i=1}^n p_i = 1, f_k(p, p\cdot x) = x \text{ for some } x \in D\}$. By the compact-valuedness and u.h.c. of the inverse demand function, we have C, C_k are compact. Because of our previous arguments, we have that there exists a compact set $K \subset \mathbb{R}^n_{++}$ that includes C and all C_k. Define $m_1 = \min_{x\in D, p\in K} p\cdot x > 0$ and $m_2 = \max_{x\in D, p\in K} p\cdot x > 0$.

We want to show that $\sup_{x\in D} |u_{f_k,\bar{p}}(x) - u_{f,\bar{p}}(x)| \to 0$ as $k \to \infty$. Suppose not. Then, there exist $\varepsilon > 0$ and a sequence (x^ℓ) in D such that $|u_{f_{k(\ell)},\bar{p}}(x^\ell) - u_{f,\bar{p}}(x^\ell)| \ge \varepsilon$, where $k(\ell)$ is increasing in ℓ. We can assume that $x^\ell \to x^*$ for some $x^* \in D$. Suppose that $x^\ell = f_{k(\ell)}(p^{k(\ell)}, m^{k(\ell)})$, where $p^{k(\ell)} \in C_{k(\ell)}$ and $m^{k(\ell)} = p^{k(\ell)}\cdot x^\ell$. Taking subsequence, we can assume that $p^{k(\ell)} \to p^* \in K$. Define $m^* = p^*\cdot x^*$. Then, $(p^{k(\ell)}, m^{k(\ell)}), (p^*, m^*) \in K\times[m_1, m_2]$, and thus

$$\begin{aligned}\|f_{k(\ell)}(p^{k(\ell)}, m^{k(\ell)}) - f(p^*, m^*)\| &\le \|f_{k(\ell)}(p^{k(\ell)}, m^{k(\ell)}) - f(p^{k(\ell)}, m^{k(\ell)})\| \\ &\quad + \|f(p^{k(\ell)}, m^{k(\ell)}) - f(p^*, m^*)\| \to 0,\end{aligned}$$

as $\ell \to \infty$. This implies that $f(p^*, m^*) = x^*$.

Now, consider the following differential equation:

$$\dot{c}(t) = I(t, c; p, m, F), c(0) = m^*,$$

where $I(t, c; p, m, F) = F((1-t)p + t\bar{p}, c + m - m^*)\cdot(\bar{p} - p)$. Let c^* be the solution of this equation with $p = p^*, m = m^*$ and $F = f$, and c_ℓ be the solution with $p = p^{k(\ell)}, m = m^{k(\ell)}$ and $F = f_{k(\ell)}$. By theorem 1, we have c^* and all c_ℓ are defined on $[0, 1]$, and $u_{f,\bar{p}}(x^*) = c^*(1)$ and $u_{f_{k(\ell)},\bar{p}}(x^\ell) = c_\ell(1) + m^{k(\ell)} - m^*$.

Choose $a > 0$ and $b > 0$, and define

$$\Pi = \{c | |c^*(t) - c| \le a \text{ for some } t \in [0, 1]\} \times \{(p, m) | \|p - p^*\| + |m - m^*| \le b\},$$

and

$$\tilde{\Pi} = \Pi \times \{F \in C^1(\mathbb{R}^n_{++}\times\mathbb{R}_{++}) | \|F - f\|_{C^1} \le b\},$$

where $\|F - f\|_{C^1}$ is the supremum value of

$$\begin{aligned}&[\|F((1-t)p + t\bar{p}, c + m - m^*) - f((1-t)p + t\bar{p}, c + m - m^*)\| \\ &+ \|DF((1-t)p + t\bar{p}, c + m - m^*) - Df((1-t)p + t\bar{p}, c + m - m^*)\|],\end{aligned}$$

where $(t, c, p, m) \in [0, 1]\times\Pi$. If a, b is chosen sufficiently small, then the following conditions hold: 1) $\Pi \subset \mathbb{R}_{++}\times\mathbb{R}^n_{++}$, 2) if $(t.c, p, m, F) \in [0, 1]\times\tilde{\Pi}$, then there exists $L > 0$ such that $\left\|\frac{\partial F}{\partial c}((1-t)p + t\bar{p}, c + m - m^*)\right\| \le L$, and 3) there exists $B > 0$ such that if $(t, c, p, m, F) \in [0, 1]\times\tilde{\Pi}$, then

$$|I(t, c; p, m, F) - I(t, c; p^*, m^*, f)| \le B[\|p - p^*\| + |m - m^*| + \|F - f\|_{C^1}].$$

Now, for any sufficiently large ℓ, we have $|m^{k(\ell)} - m^*| \le a$, $\|p^{k(\ell)} - p^*\| + |m^{k(\ell)} - m^*| \le b$ and $\|f_{k(\ell)} - f\|_{C^1} \le b$. For such ℓ, define $t_\ell = \sup\{t \in [0,1] | \forall s \in [0,t], |(c_\ell(s), p^{k(\ell)}, m^{k(\ell)}) \in \Pi\}$. Then, we have $t_\ell \ge 0$. If $t \in [0, t_\ell]$, then

$$\begin{aligned}
|c_\ell(t) - c^*(t)| &\le \int_0^t |I(\tau, c_\ell(\tau); p^{k(\ell)}, m^{k(\ell)}, f_{k(\ell)}) - I(\tau, c^*(\tau); p^*, m^*, f)| d\tau \\
&\le \int_0^t |I(\tau, c_\ell(\tau); p^{k(\ell)}, m^{k(\ell)}, f_{k(\ell)}) - I(\tau, c^*(\tau); p^{k(\ell)}, m^{k(\ell)}, f_{k(\ell)})| d\tau \\
&\quad + \int_0^t |I(\tau, c^*(\tau); p^{k(\ell)}, m^{k(\ell)}, f_{k(\ell)}) - I(\tau, c^*(\tau); p^*, m^*, f)| d\tau \\
&\le \int_0^t [L|c_\ell(\tau) - c^*(\tau)| \\
&\quad + B(\|p^{k(\ell)} - p^*\| + |m^{k(\ell)} - m^*| + \|f_{k(\ell)} - f\|_{C^1})] d\tau.
\end{aligned}$$

Now, we need the following result.

Lemma 8. For any continuous function $v : [0,t] \to \mathbb{R}$, if

$$v(s) \le \int_0^s [Av(\tau) + B] d\tau,$$

for some $A > 0$ and $B > 0$, then

$$v(s) \le \frac{B}{A}(e^{As} - 1).$$

Proof. Let $v_0 \equiv v$ and $v_{i+1}(s) = \int_0^s [Av_i(\tau) + B] d\tau$. Then, by mathematical induction we have $v_{i+1}(s) \ge v_i(s)$. Moreover, we can show that

$$|v_{i+1}(s) - v_i(s)| \le \frac{A^i C s^i}{i!}$$

for some $C > 0$, and thus v_i converges to some u uniformly. Then,

$$u(s) = \int_0^s [Au(\tau) + B] d\tau,$$

and thus $u(s) = \frac{B}{A}(e^{As} - 1)$, as desired. ■

Then, we have

$$\begin{aligned}
|c_\ell(t) - c^*(t)| &\le \frac{B(\|p^{k(\ell)} - p^*\| + |m^{k(\ell)} - m^*| + \|f_{k(\ell)} - f\|_{C^1})}{L}(e^L - 1) \\
&\equiv C(\|p^{k(\ell)} - p^*\| + |m^{k(\ell)} - m^*| + \|f_{k(\ell)} - f\|_{C^1})
\end{aligned}$$

for some $C > 0$. Now, choose any $b' \in]0, b]$ with $Cb' < a$. If ℓ is sufficiently large, then $\|p^{k(\ell)} - p^*\| + |m^{k(\ell)} - m^*| + \|f_{k(\ell)} - f\|_{C^1} \le b'$. For such ℓ, we have $t_\ell = 1$: if not, then $a \le |c_\ell(t_\ell) - c^*(t_\ell)| \le Cb' < a$, a contradiction. Then, we have

$$|c_\ell(1) - c^*(1)| \le C(\|p^{k(\ell)} - p^*\| + |m^{k(\ell)} - m^*| + \|f_{k(\ell)} - f\|_{C^1}),$$

and thus if ℓ is sufficiently large, then

$$|u_{f_{k(\ell)},\bar{p}}(x^\ell) - u_{f,\bar{p}}(x^*)| \le |c_\ell(1) - c^*(1)| + |m^{k(\ell)} - m^*| < \frac{\varepsilon}{2}.$$

Now, $|u_{f,\bar{p}}(x^\ell) - u_{f,\bar{p}}(x^*)| < \frac{\varepsilon}{2}$ if ℓ is sufficiently large. Then,

$$\begin{aligned}
\varepsilon &\le |u_{f_{k(\ell)},\bar{p}}(x^\ell) - u_{f,\bar{p}}(x^\ell)| \\
&\le |u_{f_{k(\ell)},\bar{p}}(x^\ell) - u_{f,\bar{p}}(x^*)| + |u_{f,\bar{p}}(x^*) - u_{f,\bar{p}}(x^\ell)| \\
&< \varepsilon,
\end{aligned}$$

a contradiction. Therefore, we have $u_{f_k,\bar{p}} \to u_{f,\bar{p}}$ with respect to the local uniform topology.

Now, suppose that $(x, y) \in \limsup_{k\to\infty} H(f_k)$. Then, there exists a sequence $(x_{k(\ell)}, y_{k(\ell)})$ such that $k(\ell)$ is increasing in ℓ, $u_{f_{k(\ell)},\bar{p}}(x_{k(\ell)}) \ge u_{f_{k(\ell)},\bar{p}}(y_{k(\ell)})$, and $(x_{k(\ell)}, y_{k(\ell)}) \to (x, y)$ as $\ell \to \infty$. Then by uniform convergence, we have $u_{f,\bar{p}}(x) \ge u_{f,\bar{p}}(y)$, and thus $(x, y) \in H(f)$. Next, suppose that $(x, y) \in H(f)$. Then, $u_{f,\bar{p}}(x) \ge u_{f,\bar{p}}(y)$. Choose any neighborhood U of (x, y) and $\varepsilon > 0$ such that $((1 + \varepsilon)x, y) \in U$. By theorem 2, $u_{f,\bar{p}}$ is increasing and thus $u_{f,\bar{p}}((1 + \varepsilon)x) > u_{f,\bar{p}}(y)$. Therefore, for any sufficiently large k, we have $u_{f_k,\bar{p}}((1 + \varepsilon)x) > u_{f_k,\bar{p}}(y)$, and thus $H(f_k) \cap U \ne \emptyset$. Hence, $(x, y) \in \liminf_{k\to\infty} H(f_k)$ and therefore, $\lim_{k\to\infty} H(f_k) = H(f)$. This completes the proof. ■

6.7 Proof of Theorem 5

Suppose that $\tilde{u}$ is a regular C^k-class function from Ω into $\mathbb{R}$ and $\tilde{u}(x) \ge \tilde{u}(y)$ iff $u_{f,\bar{p}}(x) \ge u_{f,\bar{p}}(y)$. Fix any $x \in \Omega$, and define

$$E(p) = \inf\{p \cdot y | \tilde{u}(y) \ge \tilde{u}(x).\}$$

In the proof of theorem 3, we showed that $E(\bar{p}) = u_{f,\bar{p}}(x)$ and if $y \equiv f(\bar{p}, E(\bar{p}))$, then $\tilde{u}(y) = \tilde{u}(x)$. Note that if $m < E(\bar{p})$, (resp. $m > E(\bar{p})$,) then $y \succ_r f(\bar{p}, m)$ (resp. $f(\bar{p}, m) \succ_r y$) and thus $\tilde{u}(y) > \tilde{u}(f(\bar{p}, m))$. (resp. $\tilde{u}(y) < \tilde{u}(f(\bar{p}, m))$.) Therefore, $u_{f,\bar{p}}(x)$ is the solution m of the following equation:

$$\tilde{u}(f(\bar{p}, m)) = \tilde{u}(x).$$

Because of the regularity of $\tilde{u}$, we can use the implicit function theorem and show that the solution of the above equation is C^k-class with respect to x. Thus, $u_{f,\bar{p}}$ is C^k-class. The regularity can be easily shown by the above equation. ■

6.8 Proof of Theorem 6

At first, we will prove (I) implies (II). Let A be the domain of f. Suppose that $(p, m) \in A$, and recall equation (3):

$$\dot{c}(t) = f((1-t)p + tq, c(t)) \cdot (q - p), c(0) = m.$$

We called the solution of above equation defined on $[0, 1]$ the connecting solution with (p, q, m). Now, we will consider the relationship between equation (3) and

$$Du(q) = f(q, u(q)), \tag{4}$$

with $u(p) = m$.

Lemma 9. Let $(p, m) \in A$ and U be an open and convex neighborhood of p. Then, there exists a solution u of (4) defined on U with $u(p) = m$ if and only if for any $q \in U$, there exists a connecting solution c_q of (3). Moreover, $u(q) = c_q(1)$, and such solution u is unique.

Proof. Suppose that there exists a solution $u : U \to \mathbb{R}_{++}$ of (4) with $u(p) = m$. Define $c_q(t) = u((1-t)p + tq)$. Then, we can easily check that c_q is the connecting solution of (3) with (p, q, m), and $c_q(1) = u(q)$.

Conversely, suppose that for any $q \in U$, there exists the connecting solution $c_q(t)$ of (3) with (p, q, m). Let $x(t; q) = c_q(t)$. By the same arguments as in the proof of lemma 6, we can show that $\frac{\partial x}{\partial q_j}(t; q) = tf^j((1-t)p+tq, x(t; q))$. Especially, $\frac{\partial x}{\partial q_j}(1; q) = f^j(q, x(1; q))$. Therefore, if we define $u(q) = x(1; q)$, then u is a solution of (4) defined on U and $u(q) = c_q(1)$.

The uniqueness of u follows at once from the uniqueness of the connecting solution. This completes the proof. ■

Lemma 10. Suppose that $x \neq y, x = f(p, m)$ and $y = f(q, w)$, and there exists a connecting solution c of (3) with (p, q, m). If $w \geq c(1)$ and there exists a connecting solution d of (3) with (q, p, w), then $p \cdot y > m$. Especially, if $w = c(1)$, then $p \cdot y > m$.

Proof. This claim can be shown by the almost same arguments as in the proof of lemma 4. ■

Lemma 11. Let C be a compact subset of A. For any $(p^*, m^*) \in C$, there exists a solution u of (4) with initial value condition $u(p^*) = m^*$ whose domain is an open ball centered at p^*, and the radius $r(C) > 0$ of this ball is independent to the choice of (p^*, m^*).

Proof. Let $\Pi = \{(w, q)|, |w - m^*| \le a, \|q - p^*\| \le b\}$. If $a, b > 0$ is sufficiently small, then $\Pi \subset A$ for any (p^*, m^*). Moreover, there exists a constant $L_1, L_2 > 0$ independent to $(p^*, m^*) \in C$ such that $|f((1-t)p^* + tq, w_1) \cdot (q - p^*) - f((1-t)p^* + tq, w_2) \cdot (q - p^*)| \le L_1|w_1 - w_2|$ and $|f((1-t)p^* + tq, m^*) \cdot (q - p^*)| \le L_2\|q - p^*\|$ if $|w_1|, |w_2| \le a, \|q\| \le b, t \in [0, 1]$. Now, let t_q^* be the supremum of $t \in [0, 1]$ such that the solution w of (3) with (p^*, q, m) can be extended to $[0, t]$ and $|w(s) - m^*| \le a$ for any $s \in [0, t]$, and let $w_q(t)$ be such a solution. Then,

$$\begin{aligned}
|w_q(t) - m^*| &\le \int_0^t |f((1-s)p^* + sq, w_q(s)) \cdot (q - p^*)|ds \\
&\le \int_0^t [|(f((1-s)p^* + sq, w_q(s)) - f((1-s)p^* + sq, m^*)) \cdot (q - p^*)| \\
&\quad + |f((1-s)p^* + sq, m^*) \cdot (q - p^*)|]ds \\
&\le \int_0^t [L_1|w_q(s) - m^*| + L_2\|q - p^*\|]ds.
\end{aligned}$$

Therefore, by lemma 8,

$$|w_q(t) - m^*| \le \frac{L_2\|q - p^*\|}{L_1}[e^{L_1} - 1].$$

Choose any b' such that $\frac{L_2 b'}{L_1}[e^{L_1} - 1] \le a$, and let $\Pi' = \{(w, q)||w - m^*| \le a, \|q - p^*\| \le b'\}$. Because w_q cannot be defined on $[0, 1]$ only if there exists $t \in [0, 1[$ such that $(w_q(t), q) \notin \Pi'$, we have w_q can be defined on $[0, 1]$ if $\|q - p^*\| \le b'$. By lemma 9, we can choose $r(C) = b'$. ■

Lemma 12. Choose any $x \in \Omega$ and suppose that C is a compact subset A including $\{(p, p \cdot x)|p \in G(x)\}$. Then, there exists a solution $u : U \to \mathbb{R}_{++}$ of (4) such that $p \cdot x = u(p)$ for any $p \in G(x)$, and the domain U is an open and convex neighborhood of $G(x)$ including $\{q|\exists p \in G(x), \|q - p\| < r(C)\}$.

Proof. Choose any $p \in G(x)$. By lemma 11, there exists a solution $u_p : U_p \to \mathbb{R}^n_{++}$ such that $u_p(p) = p \cdot x$ and U_p is an open ball of radius $r(C)$ centered at p. Now, let $U_p \cap U_q \neq \emptyset$ for some $p, q \in G(x)$. Then, there exists $r \in [p, q] \cap U_p \cap U_q$. Note that $r \in G(x)$. We will show that $u_p(r) = r \cdot x =$

$u_r(r)$. Let $p(t) = (1-t)p + tr$ and $d(t) = (1-t)p \cdot x + tr \cdot x$. By assumption, we have $p(t) \in G(x)$ and thus $f(p(t), d(t)) = x$. Therefore,

$$\dot{d}(t) = f(p(t), d(t)) \cdot (r - p), d(0) = p \cdot x.$$

Meanwhile, define $c(t) = u_p(p(t))$. Then,

$$\dot{c}(t) = f(p(t), c(t)) \cdot (r - p), c(0) = p \cdot x.$$

Therefore, we have $d \equiv c$ and thus $r \cdot x = d(1) = c(1) = u_p(r)$. Symmetrically, we can show that $u_q(r) = r \cdot x$. By lemma 9, $u_p(p') = u_q(p')$ for any $p' \in U_p \cap U_q$, and thus we can define $u(q) = u_p(q)$ for any $q \in U_p$ with $p \in G(x)$. Clearly the domain U of u includes $\{q | \exists p \in G(x), \|q - p\| < r(C)\}$ and u is a solution of (4) with $u(p) = p \cdot x$ for any $p \in G(x)$. ■

Lemma 13. Choose any $x, v \in \Omega$ such that x is not proportional to v, and define the followings:[23]

$$\begin{aligned}
a_1 &= \frac{1}{\|x\|} x, \\
a_2 &= \frac{1}{\|v - \frac{v \cdot x}{\|x\|^2} x\|} (v - \frac{v \cdot x}{\|x\|^2} x), \\
Py &= (a_1 \cdot y) a_1 + (a_2 \cdot y) a_2, \\
Ry &= (a_1 \cdot y) a_2 - (a_2 \cdot y) a_1, \\
v_1 &= \arg\min\{w \cdot a_1 | w \in P\mathbb{R}^n_+, \|w\| = 1, w \cdot a_2 \geq 0\}, \\
v_2 &= \arg\min\{w \cdot a_1 | w \in P\mathbb{R}^n_+, \|w\| = 1, w \cdot a_2 \leq 0\}, \\
y_1 &= \{s_1 v | s_1 \in \mathbb{R}\} \cap \{x + s_2 R v_1 | s_2 \in \mathbb{R}\}, \\
y_2 &= \{s_3 v | s_3 \in \mathbb{R}\} \cap \{x + s_4 R v_2 | s_4 \in \mathbb{R}\}, \\
\Delta &= \{w \in \text{span}\{x, v\} | w \cdot Rv \leq 0,\ w \cdot v_1 \geq x \cdot v_1,\ w \cdot v_2 \leq x \cdot v_2\}.
\end{aligned}$$

Moreover, we define $y_1 = y_2 = x$ if x is proportional to v. Then, the following properties hold.

1) a_1, a_2 is an orthonormal basis of $\text{span}\{x, v\}$ derived by the Gram-Schmidt method, and P is an orthogonal projection from $\mathbb{R}^n$ onto $\text{span}\{x, v\}$.

2) R is an orthogonal transform on $\text{span}\{x, v\}$ such that $y \cdot Ry = 0$ for any $y \in \text{span}\{x, v\}$. Moreover, $R^2 y = -y$ and $R^3 = R^{-1} = -R$. If T is another orthogonal transform on $\text{span}\{x, v\}$ and $y \cdot Ty = 0$ for any $y \in \text{span}\{x, v\}$, then $T = R$ or $T = -R$.

[23] We sometimes abbreviate $P(x, v)$ as P, $R(x, v)$ as R, and so on.

3) $P(\mathbb{R}^n_+)$ is a convex cone generated by v_1, v_2. Moreover, v_1, v_2 is continuous in (x, v).

4) y_1, y_2 is a single-valued continuous function from Ω^2 into Ω. Moreover, both are proportional to v and $y_2 \geq y_1$.

5) $\Delta = \mathrm{co}\{x, y_1, y_2\} = RP(\mathbb{R}^n_+) \cap \{y \in \mathrm{span}\{x, v\} | y \cdot v \leq 0\}$. Hence, Δ is a compact set in Ω.

Proof: See the proof of theorem 1 of Hosoya (2013). ■

Now, choose any $x, v \in \Omega$ such that x is not proportional to v. Define $x(t) = (1-t)x + tv$. We call a curve $c : I \to \mathrm{span}\{x, v\}$ a **indifference curve with parameter** (x, v) if the following statements hold:

i) I is an interval, $I \subset [0, 1]$ and $0 \in I$.

ii) $c(t) \in \Delta(c(s), v)$ for any $s, t \in I$ with $s < t$, $c(0) = x$, and $c(t) \cdot Rv = (1-t)x \cdot Rv$.

iii) There exists a family of the solutions $(u_t)_{t \in I}$ of (4) such that the domain U_t of u_t is an open and convex neighborhood of $G(c(t))$, $p \cdot c(t) = u_t(p)$ for any $p \in G(c(t))$, and $u_s(p) = u_t(p)$ if $p \in U_s \cap U_t$.

Moreover, if $I = [0, 1]$, then we say that this indifference curve is **maximal**.

Lemma 14. For any $x, v \in \Omega$ such that x is not proportional to v, there uniquely exists a maximal indifference curve $c : [0, 1] \to \Omega$ with parameter (x, v). Moreover, $c(t)$ is continuous and $p \cdot c(t) > p \cdot c(s)$ for any $s, t \in [0, 1]$ with $s \neq t$ and $p \in G(c(s))$.

Proof. In this proof, we abbreviate $R(x, v)$ as R, $v_i(x, v) = v_i$, and so on. Note that for any $y, z \in \mathrm{span}\{x, v\} \cap \Omega$ such that y is not proportional to z, we have $P(y, z) = P$, either $R(y, z) = R$ or $R(y, z) = -R$, and $R(y, z) = -R(z, y)$. Also, by Cauchy-Schwarz inequality, we can show that $x \cdot Rv < 0$ and $v \cdot Rx > 0$. Further, we can show that if $y \in \Delta$ and y is not proportional to v, then $\Delta(y, v) \subset \Delta$.

Let $C = \cup_{y \in \Delta}\{(p, p \cdot y) | p \in G(y)\}$. By u.h.c. of G, we have C is a compact set in A and thus $r(C)$ can be defined.

By lemma 12, there exists a solution $u : U \to \mathbb{R}^n_{++}$ of (4) such that U is an open and convex neighborhood of $G(x)$ and $u(p) = p \cdot x$ for any $p \in G(x)$. We assume that $U \subset \{q | \exists p \in G(x), \|q - p\| < r(C)\}$. Now, define $x_i(t) = (1-t)x + ty_i$ for $i = 1, 2$. Then, by u.h.c. of G, there exists $\bar{t} > 0$ such that $G(y) \subset U$ for any $y \in [x_1(t), x_2(t)]$ with $t \in [0, \bar{t}]$. For any $p \in \mathbb{R}^n_{++}$, let $q = \frac{1}{\|Pp\|}Pp$. Then, $q = c_1 a_1 + c_2 a_2$ for some c_1, c_2 with $c_1^2 + c_2^2 = 1$. We have $c_i = q \cdot a_i$, and thus $c_1 > 0$, and either $c_2 < 0$ or $c_1 \in [v_1 \cdot a_1, 1]$ holds. In both cases,

$$Rv_1 \cdot q = (v_1 \cdot a_1)c_2 - (v_1 \cdot a_2)c_1 \leq 0.$$

Similarly, we can show that

$$Rv_2 \cdot q \geq 0.$$

Therefore, we have $p \cdot x_1(t) \leq p \cdot x$ for any $p \in G(x)$ and $q \cdot x_2(t) \geq q \cdot x$ for any $q \in G(x_2(t))$.

Suppose that $q \cdot x_1(t) \geq u(q)$ for some $q \in G(x_1(t))$. By lemma 12, we have there exists a solution $v : V \to \mathbb{R}^n_{++}$ of (4) such that V is an open and convex neighborhood of $G(x_1(t))$, $q \cdot x_1(t) = v(q)$ and there exists $p \in V \cap G(x)$. Then, by lemma 10, we have $p \cdot x_1(t) > p \cdot x$, which is absurd. Therefore, we have $q \cdot x_1(t) < u(q)$ for any $q \in G(x_1(t))$. By the same arguments, we can show that $q \cdot x_2(t) > u(q)$ for any $q \in G(x_2(t))$.

Define $y(s) = (1-s)x_1(t) + sx_2(t)$, and suppose that there is no $s \in [0, 1]$ such that there exists $q \in G(y(s))$ with $u(q) = q \cdot y(s)$. Let $s^* = \sup\{s \in [0, 1] | \forall q \in G(y(s)), q \cdot y(s) < u(q)\}$. Then, there is a sequence s_k^1, s_k^2 such that $s_k^1 \uparrow s^*, s_k^2 \downarrow s^*$, $q \cdot y(s_k^1) < u(q)$ for any $q \in G(y(s_k^1))$, and $q \cdot y(s_k^2) > u(q)$ for any $q \in G(y(s_k^2))$. Choose any $q^{1,k}, q^{2,k}$ with $q^{i,k} \in G(y(s_k^i))$. Taking subsequence, we can assume that $q^{i,k} \to q^i \in G(y(s^*))$. By continuity of u, we have $q^1 \cdot y(s^*) \leq u(q^1)$ and $q^2 \cdot y(s^*) \geq u(q^2)$, and thus there exists $q \in [q^1, q^2]$ such that $q \cdot y(s^*) = u(q)$. However, by the C axiom, we have $q \in G(y(s^*))$, a contradiction. By lemma 12, there exists a solution $v : V \to \mathbb{R}^n_{++}$ of (4) such that V is an open and convex neighborhood of $G(y)$ and $r \cdot y = v(r)$ for any $r \in G(y)$. Then, $u(q) = q \cdot y = v(q)$, and by lemma 9, we have $u(r) = v(r) = r \cdot y$ for any $r \in G(y)$. Therefore, we have there exists $y = y(s^*) \in [x_1(t), x_2(t)]$ such that $q \cdot y = u(q)$ for any $q \in G(y)$. Choose any $s \in [0, s^*[$. If $u(p') = p' \cdot y(s)$ for some $p' \in G(y(s))$, then $p' \cdot y(s^*) > u(p')$ and by lemma 10, $q' \cdot y(s^*) > u(q')$ for $q' \in G(y(s^*))$, a contradiction. Therefore, such s^* is unique. Define $c(t) = y$. Then we get a curve $c : [0, \bar{t}] \to \text{span}\{x, v\}$. Note that $c(t) \cdot Rv = x_i(t) \cdot Rv = (1-t)x \cdot Rv + ty_i \cdot Rv = (1-t)x \cdot Rv$. Let $t, s \in [0, \bar{t}]$ and suppose $s < t$. If $c(t) \notin \Delta(c(s), v)$, then by the same arguments as above, we have either $p \cdot c(t) \leq p \cdot c(s)$ for some $p \in G(c(s))$ or $q \cdot c(s) \leq q \cdot c(t)$ for some $q \in G(c(t))$. If the former holds, then $u(p) = p \cdot c(s) \geq p \cdot c(t)$ and $c(s) \neq c(t)$, and by lemma 10 we have $u(p') = p' \cdot c(t) < u(p')$ for $p' \in G(c(t))$, a contradiction. If the latter holds, then $u(q) = q \cdot c(t) \geq q \cdot c(s)$ and $c(s) \neq c(t)$, and by lemma 10 we have $u(q') = q' \cdot c(s) < u(q')$ for $q' \in G(c(s))$, a contradiction. Therefore, we have $c(t) \in \Delta(c(s), v)$ and this curve is an indifference curve with parameter (x, v). (Clearly, we can put $u_t \equiv u$.)

Let $c : I \to \mathrm{span}\{x, v\}$ be some indifference curve with parameter (x, v). By definition of the indifference curve, we have $c(t) \in [x_1^*(t), x_2^*(t)]$ if $s, t \in I, s < t$ and $t - s$ is sufficiently small, where $x_i^*(t) = \frac{t-s}{1-s}c(s) + \frac{1-t}{1-s}y_i(c(s), v)$. Because $x_i^*(t) \to c(s)$ as $t \downarrow s$, we have $c(t) \to c(s)$. Similarly, we can show that $c(t) \to c(s)$ if $t \uparrow s$. Hence, $c(t)$ is continuous.

Suppose $c : [0, t] \to \mathrm{span}\{x, v\}$ and $d : [0, t] \to \mathrm{span}\{x, v\}$ are two indifference curves with parameter (x, v). We will show that $c \equiv d$. Let $T' = \{t \in [0, 1] | \forall s \in [0, t], d(s) = c(s)\}$. We have $0 \in T'$ and by continuity of $c(t)$ and $d(t)$, T' is a closed subset of $[0, 1]$. Define $t^* = \sup T'$ and suppose $t^* \neq t$. Because T' is closed, we have $c(t^*) = d(t^*)$. By definition of the indifference curve and by lemma 9, there exists a solution $u : U \to \mathbb{R}_{++}$ such that U is an open and convex neighborhood of $G(c(t^*))$ and there exists $s^* \in [t^*, t]$ such that if $s \in [t^*, s^*]$, then $G(c(s)) \subset U, G(d(s)) \subset U, p \cdot c(s) = u(p)$ for any $p \in G(c(s))$, and $q \cdot d(s) = u(q)$ for any $q \in G(d(s))$. Choose any $s \in]t^*, s^*]$ such that $c(s) \neq d(s)$, and choose any $p \in G(c(s)), q \in G(d(s))$. Because $c(s), d(s) \in [x_1(s), x_2(s)]$, there exists a number $\alpha \neq 0$ such that $d(s) = c(s) + \alpha v$. Without loss of generality, we assume that $\alpha > 0$. Choose any $p \in G(c(s))$ and $q \in G(d(s))$. Then, we have $p \cdot d(s) = p \cdot c(s) + \alpha p \cdot v > p \cdot c(s) = u(p)$. Therefore, by lemma 10, $u(q) = q \cdot d(s) > u(q)$, a contradiction. Thus, we have $t^* = t$, and hence $c(t) \equiv d(t)$. Especially, if there exists a maximal indifference curve $c : [0, 1] \to \mathrm{span}\{x, v\}$, then such a curve is unique, and $c(t)$ is continuous.

Next, let $c : I \to \mathrm{span}\{x, v\}$ be an indifference curve with parameter (x, v) and $s, t \in I$ with $s < t$. We will show that $c(t)$ is not proportional to $c(s)$ and $R(c(s), c(t)) = R$. At first, recall $c(t) \in \Delta(c(s), v)$. Because $c(s) \neq c(t)$, there exists $p \in \mathbb{R}^n_+ \setminus \{0\}$ such that $c(t) = c(s) + RPp$. Then, $Rc(s) \cdot c(t) = c(s) \cdot p > 0$, and thus $c(t)$ is not proportional to $c(s)$. Define $R^* = R(c(s), c(t))$. By lemma 13, we have either $R^* = R$ or $R^* = -R$. However, by definition we have $R^* c(s) \cdot c(t) > 0$, and thus $R = R^*$. Thus, our claim is correct.

Now, let $c : I \to \mathrm{span}\{x, v\}$ be an indifference curve with parameter (x, v), and $s, t \in I$ with $s < t$. We will show that $p \cdot c(t) > p \cdot c(s)$ for any $p \in G(c(s))$ and $q \cdot c(s) > q \cdot c(t)$ for any $q \in G(c(t))$. By definition, there exists a family of solutions $(u_t)_{t \in [s,t]}$ of (4) such that the domain $U_{t'}$ of $u_{t'}$ is an open and convex neighborhood of $G(c(t'))$, $p \cdot c(t') = u_{t'}(p)$ for any $p \in G(c(t'))$, and $u_{t'}(p) = u_{s'}(p)$ if $p \in U_{t'} \cap U_{s'}$. Let $V_{t'} = \{s' \in [s, t] | G(c(s')) \subset U_{t'}\}$. By u.h.c. of G, $(V_{t'})$ is an open covering of $[s, t]$, and thus there exists a finite subcovering (V_{t_i}). We can show by mathematical induction on the number of V_{t_i} that there exists $(x_i)_{i=0,\ldots,k}$ such that $x_i = c(s_i)$ with $s = s_0 <$

$s_1 < ... < s_\ell = t$ and $G(x_i) \cup G(x_{i+1}) \subset U_{t_j}$ for some j. We will show by mathematical induction on k that $p \cdot c(s_i) > p \cdot c(s_0)$ and $q \cdot c(s_i) < q \cdot c(s_0)$ for any $p \in G(c(s_0)), q \in G(c(s_i)), i = 1, ..., k$.

If $k = 1$, then $G(c(s_0)) \cup G(c(s_1)) \subset U_{t_j}$, and by lemma 10, we have $p \cdot c(s_1) > p \cdot c(s_0)$ and $q \cdot c(s_1) < q \cdot c(s_0)$ for any $p \in G(c(s_0)), q \in G(c(s_1))$.

If the claim is correct for $k-1$, then $p \cdot c(s_i) > p \cdot c(s_0), q \cdot c(s_i) < q \cdot c(s_0)$ for any $p \in G(c(s_0)), q \in G(c(s_i)), i = 1, ..., k-1$. Also, by lemma 10, $q \cdot c(s_k) > q \cdot c(s_{k-1}), r \cdot c(s_k) < r \cdot c(s_{k-1})$ for any $q \in G(c(s_{k-1})), r \in G(c(s_k))$. Now, choose any $p \in G(c(s_0)), q \in G(c(s_{k-1}))$. Then, there exists $t_1, t_2 > 0$ and $t_3, t_4 < 0$ such that $c(s_{k-1}) + t_1 RPp, c(s_{k-1}) + t_2 RPq$ are proportional to $c(s_k)$ and $c(s_{k-1}) + t_3 RPp, c(s_{k-1}) + t_4 RPq$ are proportional to $c(s_0)$.[24]Now,

$$p \cdot (c(s_{k-1}) + t_3 RPp) = p \cdot c(s_{k-1}) > p \cdot c(s_0),$$

and thus $c(s_{k-1}) + t_3 RPp \gg c(s_0)$. Similarly,

$$q \cdot (c(s_{k-1}) + t_4 RPq) = q \cdot c(s_{k-1}) < q \cdot c(s_0),$$

and thus $c(s_{k-1}) + t_4 RPq \ll c(s_0)$. This implies that

$$p \cdot (c(s_{k-1}) + t_4 RPq) < p \cdot (c(s_{k-1}) + t_3 RPp) = p \cdot c(s_{k-1}),$$

and thus $p \cdot RPq > 0$. Therefore,

$$p \cdot (c(s_{k-1}) + t_2 RPq) > p \cdot c(s_{k-1}) = p \cdot (c(s_{k-1}) + t_1 RPp),$$

and we obtain $c(s_{k-1}) + t_2 RPq \gg c(s_{k-1}) + t_1 RPp$. Meanwhile,

$$q \cdot c(s_k) > q \cdot c(s_{k-1}) = q \cdot (c(s_{k-1}) + t_2 RPq),$$

and thus $c(s_k) \gg c(s_{k-1}) + t_2 RPq$. Therefore,

$$p \cdot c(s_k) > p \cdot (c(s_{k-1}) + t_1 RPp) = p \cdot c(s_{k-1}) > p \cdot c(s_0).$$

By the same arguments, we can show that $r \cdot c(s_0) > r \cdot c(s_k)$ for any $r \in G(c(s_k))$. Therefore, our claim is correct. Especially, $G(c(s)) \cap G(c(t)) = \emptyset$ if $s \neq t$.

The remaining claim is the existence of a maximal indifference curve. Let T be the set of all $\bar{t} \in [0, 1]$ such that there exists an indifference curve $c : [0, \bar{t}] \to \mathrm{span}\{x, v\}$. We had already shown that T includes a neighborhood

[24] If, for example, such $t_1 \in \mathbb{R}$ does not exist, then RPp is proportional to v, and $0 = Rv \cdot RPp = v \cdot p > 0$, a contradiction. Also, because $R(c(s_{k-1}), c(s_k)) = R$, we have $Rc(s_k) \cdot c(s_{k-1}) < 0$, and thus $t_1 > 0$.

of 0. Moreover, if $0 < s < t$ and $t \in T$, then $s \in T$. Further, by repeating the proof T includes a neighborhood of 0, we can easily show that T is open. It suffices to show that $t^* \equiv \sup T \in T$.

Now, let (t_k) be a sequence in T such that $t_k \uparrow t^*$ such that $c^{t_k}(t_k)$ converges to $y \in [x_1(t^*), x_2(t^*)]$, where $c^{t_k} : [0, t_k] \to \mathrm{span}\{x, v\}$ is an indifference curve with parameter (x, v). Note that because $c^{t_k}(t_k) \in \Delta$, we can choose such a sequence. By our previous arguments, we have $c^{t_k}(t) = c^{t_{k'}}(t)$ if both can be defined. Define $c(t) = c^{t_k}(t)$ if $t \in [0, t_k]$ for some k. Note that $c(t) \in \Delta(c(s), v)$ for any $t, s \in [0, t^*[$ with $s < t$. If y' is another limit point of $c(t)$ with $t \uparrow t^*$, then we have $y' \notin \Delta(c(t_k), v)$ for some k, and thus $c(t) \notin \Delta(c(t_k), v)$ for some $t \in [t_k, t^*[$, a contradiction. Therefore, we have $c(t) \to y$ as $t \to t^*$.

Let $v : V \to \mathbb{R}$ be a solution of (4) such that $V = \{p | \exists q \in G(y), \|p - q\| < r(C)\}$ and $v(p) = p \cdot y$ for any $p \in G(y)$. Now, we will show that there exists $t^+ \in [0, t^*[$ such that $G(c(t)) \cap V \neq \emptyset$ and $v(p) = p \cdot c(t)$ for any $p \in G(c(t)) \cap V$ with $t \in [t^+, t^*[$. Suppose not. Then, there exists a sequence (s^k) such that $s^k \uparrow t^*$ and $v(p^k) \neq p^k \cdot c(s^k)$ for some $p^k \in G(c(s_k))$. Taking subsequence, we can assume that $p^k \to p^* \in G(y)$. Then, $p^k \cdot c(s_k) \to p^* \cdot y$. Now, let $u_k : U_k \to \mathbb{R}^n_{++}$ be the solution of (4) such that U_k is an open ball with radius $r(C)$ centered at p^k, and $u_k(p^k) = p^k \cdot c(s_k)$. Then, $p^* \in U_k$ if k is sufficiently large. Define $c_k(t) = u_k((1-t)p^k + tp^*)$. Then,

$$\dot{c}_k(t) = f((1-t)p^k + tp^*, c_k(t)) \cdot (p^* - p^k), c(0) = p^k \cdot c(s_k).$$

By continuity of the solution of ordinary differential equation with parameter and initial value, we have $u_k(p^*) = c_k(1) \to p^* \cdot y$.

Now, let $p \in V$ and suppose there exists $q \in G(y)$ with $\|p - q\| \le \frac{r(C)}{2}$. Then, the following differential equation

$$\dot{c}(t) = f((1-t)p^* + tp, c(t)) \cdot (p - p^*)$$

have a solution $c_p : [0, 1] \to \mathbb{R}_{++}$ with $c_p(0) = p^* \cdot y$, namely, $c_p(t) = v((1-t)p^* + tp)$. Again by continuity of the solution of ordinary differential equation, there exists $\varepsilon > 0$ such that if $\max\{\|p - p'\|, |p^* \cdot y - m'|\} < \varepsilon$, then there exists a solution $d_{p'} : [0, 1] \to \mathbb{R}_{++}$ of

$$\dot{d}_{p'}(t) = f((1-t)p^* + tp', d_{p'}(t)) \cdot (p' - p^*),$$

with $d_{p'}(0) = m'$. Because the set $\bar{V} = \{p \in V | \exists q \in G(y), \|q - p\| \le \frac{r(C)}{2}\}$ is compact, we can choose such $\varepsilon > 0$ uniformly on this set. If k is sufficiently large, then $|u_k(p^*) - p^* \cdot y| < \varepsilon$, and by lemma 9, $\bar{v} : p \mapsto d_p(1)$ is a solution of (4) with $\bar{v}(p^*) = u_k(p^*)$. We assume that k is so large that $G(c(t)) \subset \bar{V}$ if

$t \in [s_k, t^*[$. Now, let $T^* = \{t \in [s_k, t^*[|\forall s' \in [s_k, t], \forall p \in G(c(s')), p \cdot c(s') = \bar{v}(p)\}$. Because $p^k \cdot c(s_k) = u_k(p^k) = \bar{v}(p^k)$, we have $s_k \in T^*$. By continuity of $\bar{v}$ and c, we have T^* is closed in $[s_k, t^*[$. It is easy to show that T^* is open. Therefore, we have $T^* = [s_k, t^*[$ and $\bar{v}(p^{k'}) = p^{k'} \cdot c(s_{k'})$ for sufficiently large k'. Then, $\bar{v}(p^*) = p^* \cdot y = v(p^*)$. This implies that $p^k \cdot c(s_k) = \bar{v}(p^k) = v(p^k)$, a contradiction.

Then, there exists such a t^+. Now, let $r^* = \min_{t \in [0,t^+], p \in G(c(t)), q \in G(y)} \|q - p\|$. By the same arguments as above, we can show that $G(y) \cap G(c(t)) = \emptyset$ for all $t \in [0, t^*[$. Therefore, we have $r^* > 0$. Define $\tilde{V} = \{p | \exists q \in G(y), \|q - p\| < \frac{r^*}{2}\}$, and $\tilde{v}$ as the restriction of v to $\tilde{V}$. Now, choose any $t \in [t^+, t^*[$ such that $G(c(s)) \subset \tilde{V}$ for any $s \in [t, t^*[$, and define $c(t^*) = y$ and a family of solutions $(u_s)_{s \in [0,t^*]}$ of (4) as follows. If $s \in [0, t]$, then there is a family of solutions $(u_r^t)_{r \in [0,t]}$ of (4) such that $q \cdot c(r) = u_r^t(q)$ for any $q \in G(c(r))$ with $r \in [0, t]$ and $u_{r_1}^t(r) = u_{r_2}^t(r)$ if both values can be defined. Put $u_s = u_s^t$. If $s \in [0, t^+]$, then restrict u_s^t to some open and convex neighborhood U_s of $G(c(s))$ such that $U_s \cap \tilde{V} = \emptyset$. If $s \in]t, t^*[$, then put $u_s = \tilde{v}$. It is easy to show that $(u_s)_{s \in [0,t^*]}$ has the required condition in the definition of the indifference curve, and thus we have $t^* \in T$. This completes the proof. ■

Lemma 15. Let $x, v \in \Omega$ and x be not proportional to v. Let $c : [0, 1] \to \text{span}\{x, v\}$ be the maximal indifference curve with parameter (x, v), and $d : [0, 1] \to \text{span}\{x, v\}$ be the maximal indifference curve with parameter (v, x). Then, $d(1) \leq x$ if and only if $c(1) \geq v$.

Proof. At first, we shall show that there exists a continuous increasing function $\alpha : [0, 1] \to [0, 1]$ such that $c(\alpha(1 - t))$ is the maximal indifference curve with parameter $(c(1), x)$. We abbreviate $R(x, v)$ as R. Note that by lemma 13, we have $R(c(1), x) = -R$.

Let $c(t_1) \cdot Rx = c(t_2) \cdot Rx$ for some $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$. Clearly we have $c(t_1) \neq c(t_2)$, and thus there exists $\gamma \neq 0$ such that $c(t_2) = c(t_1) + \gamma x$. If $\gamma > 0$, then $p \cdot c(t_2) > p \cdot c(t_1) > p \cdot c(t_2)$ for any $p \in G(c(t_2))$ by lemma 14, a contradiction. If $\gamma < 0$, then $p \cdot c(t_1) > p \cdot c(t_2) > p \cdot c(t_1)$ for any $p \in G(c(t_1))$ by lemma 14, a contradiction. Therefore, we have such t_1, t_2 do not exist, and there uniquely exists a function $\alpha : [0, 1] \to [0, 1]$ such that $c(\alpha(1 - t)) \cdot Rx = (1 - t)v \cdot Rx$. Now, the continuity of α can be easily shown, and then clearly α is increasing. If $c(\alpha(1-t)) \notin \Delta(c(\alpha(1-s)), x)$ for some $s, t \in [0, 1]$ with $s < t$, then by the similar arguments as in the proof of lemma 14, we have either $p \cdot c(\alpha(1-t)) \leq p \cdot c(\alpha(1-s))$ for $p \in G(c(\alpha(1-s)))$ or $p \cdot c(\alpha(1 - s)) \leq p \cdot c(\alpha(1 - t))$ for $p \in G(c(\alpha(1 - t)))$, which contradicts lemma 14 itself. Also, it is clear that $\alpha(0) = 1, \alpha(1) = 0$. Therefore, our claim holds.

If $c(1) = v$, then $d(t) = c(\alpha(1-t))$, and thus $d(1) = c(0) = x$. Next, suppose that $c(1) \gg v$. Then, there exists a continuous function $\beta : [0,1] \to \mathbb{R}$ such that $d(t) = c(\alpha(1-t)) + \beta(t)x$. By assumption, we have $\beta(0) < 0$. If $\beta(1) \geq 0$, then there exists t^* such that $\beta(t^*) = 0$. Then, by easy arguments, we have the trajectory of $c(t)$ is the same as of $d(t)$. Hence $c(1) = d(0) = v$, a contradiction. Hence, we have $\beta(1) < 0$ and thus $d(1) \ll x$.

Similarly, if $c(1) \ll v$, then $d(1) \gg x$. This completes the proof. ■

Lemma 16. Choose any linearly independent $x, v, z \in \Omega$, and let $c_1(t)$ be the maximal indifference curve with parameter (x, v), $c_2(t)$ be the maximal indifference curve with parameter (x, z) and $c_3(t)$ be the maximal indifference curve with parameter $(c_1(1), z)$. Then, $c_2(1) = c_3(1)$.

Proof. Let $c^t(s)$ be the maximal indifference curve with parameter $(x, c_3(t))$, and $T = \{\bar{t} \in [0,1] | \forall t \in [0, \bar{t}], c_3(t) = c^t(1)\}$. We have $0 \in T$, and if $0 < s < t$ and $t \in T$, then $s \in T$. It suffices to show that T is open and closed in $[0,1]$.

At first, suppose that $t \in T$ and $t < 1$. Let $(u_s)_{s\in[0,1]}$ be a corresponding family of solutions of (4) such that the domain U_s of u_s is an open and convex neighborhood of $G(c_t(s))$, $u_s(p) = p \cdot c_t(s)$ if $p \in G(c_t(s))$, and $u_{s_1}(p) = u_{s_2}(p)$ if both values are defined. Let $B_r = \{z \in \mathbb{R}^n | \|z\| < r\}$ and

$$r(s) = \sup[\{r | \exists s' \in [0,1], G(c_t(s) + B_r) \subset U_{s'}\}.$$

Then, we can easily show that if $r^* > 0$ is sufficiently small, then $r^* < r(s)$ for any $s \in [0,1]$. Now, let $(v_{t'})_{t'\in[0,1]}$ be a family of solutions of (4) such that the domain $V_{t'}$ is an open and convex neighborhood of $G(c_3(t'))$, $v_{t'}(p) = p \cdot c_3(t')$ for any $p \in G(c_3(t'))$, and $v_{t_1}(p) = v_{t_2}(p)$ if both are defined. Without loss of generality, we assume that $G(c_t(1) + B_{r^*}) \subset V_t$.

Now, let $x_i(t', s) = (1-s)x + s y_i(x, c_3(t'))$. Then, $c_{t'}(s) \in [x_1(t', s), x_2(t', s)]$ and by using lemma 14, we can show that $x_1(t, s) \ll c_t(s) \ll x_2(t, s)$ if $s > 0$. Because x_i is continuous in (t', s), there exists $\varepsilon > 0$ such that if $t' > t$ and $t' - t < \varepsilon$, then $x_1(t', s) \ll c_t(s) \ll x_2(t', s)$ for any $s \in]0,1]$. Therefore, $p \cdot x_1(t', s) < p \cdot c_t(s)$ for any $p \in G(x_1(t', s))$, and $q \cdot c_t(s) < q \cdot x_2(t', s)$ for any $q \in G(x_2(t', s))$. Define $y(t', s, s') = (1-s')x_1(t', s) + s' x_2(t', s)$, and let $s_1(t', s) = \sup\{\bar{s} | \forall s' \in [0, \bar{s}], \forall p \in G(y(t', s, s')), p \cdot y(t', s, s') < p \cdot c_t(s)\}$ and $s_2(t', s) = \inf\{\bar{s} | \forall s' \in [\bar{s}, 1], \forall p \in G(y(t', s, s')), p \cdot y(t', s, s') > p \cdot c_t(s)\}$. We will show that $\sup_{s\in]0,1]} \|y(t', s, s_i(t', s)) - c_t(s)\| \to 0$ as $t' \to t$. We treat only the case $i = 1$. If not, then there exists a sequence (t_k) and (s_k) such that $t_k \downarrow t$ and $\|y(t_k, s_k, s_1(t_k, s_k)) - c_t(s_k)\| > \varepsilon'$ for some $\varepsilon' > 0$. Taking subsequence, we can assume that $s_k \to s^*$ and $y(t_k, s_k, s_1(t_k, s_k)) \to z$. Because $y(t_k, s_k, s_1(t_k, s_k)) \in [x_1(t_k, s_k), x_2(t_k, s_k)]$, we have $z \in [x_1(t, s^*), x_2(t, s^*)]$ and $\|z - c_t(s^*)\| \geq \varepsilon'$. Therefore, either $z \ll c_t(s^*)$ or $c_t(s^*) \ll z$. If the former holds, then $c_t(s_k) \gg y(t_k, s_k, s_1(t_k, s_k))$ for some k, and thus $c_t(s_k) \gg y(t_k, s_k, s')$ for any $s' > s_1(t_k, s_k)$ such that $s' - s_1(t_k, s_k)$ is sufficiently small. Then $p \cdot c_t(s_k) > p \cdot y(t_k, s_k, s')$ for any $p \in G(y(t_k, s_k, s'))$, a contradiction. If the latter holds, then $y(t_k, s_k, s_1(t_k, s_k)) \gg c_t(s_k)$ for some k, and thus $y(t_k, s_k, s') \gg c_t(s_k)$ for some $s' < s_1(t_k, s_k)$. For such s', we have $p \cdot c_t(s_k) < p \cdot y(t_k, s_k, s')$ for any $p \in G(y(t_k, s_k, s'))$, a contradiction. Therefore, our claim holds, and without loss of generality, we can assume that $\|y(t', s, s_i(t', s)) - c_t(s)\| < r^*$ for any $t' > t$ with $t' - t < \varepsilon$.

Define $d(0) = x$. Next, suppose that $s > 0$. By definition of r^*, there exists $s' \in [0, 1]$ such that $G(c_t(s) + B_{r^*}) \subset U_{s'}$. By u.h.c. of G, $G(y(t', s, s'')) \subset U_{s'}$ if $s'' < s_1(t', s)$ and $s_1(t', s) - s''$ is sufficiently small. Choose any $p \in G(y(t', s, s''))$ and $q \in G(c_t(s))$. We have $u_{s'}(q) = q \cdot c_t(s)$. If $u_{s'}(p) < p \cdot y(t', s, s'')$, then we have $p \cdot c_t(s) > p \cdot y(t', s, s'') > p \cdot f(p, u_{s'}(p))$, which contradicts lemma 1.[25] Therefore, we have $u_{s'}(p) \geq p \cdot y(t', s, s'')$. By u.h.c. of G, there exists $p_1 \in G(y(t', s, s_1(t', s)))$ such that $u_{s'}(p_1) \geq p_1 \cdot y(t', s, s_1(t', s))$. Symmetrically, we can show that there exists $p_2 \in G(y(t', s, s_2(t', s)))$ such that $p_2 \cdot y(t', s, s_2(t', s)) \geq u_{s'}(p_2)$. By repeating our previous arguments in the proof of lemma 14, we can show that there uniquely exists $d(s) \in [y(t', s, s_1(t', s)), y(t', s, s_2(t', s))]$ such that $u_{s'}(p) = p \cdot d(s)$ for any $p \in G(d(s))$. We will show that $d(s) = c_{t'}(s)$.

Let $S^* = \{\bar{s} \in [0, 1] | \forall s \in [0, \bar{s}], d(s) = c_{t'}(s)\}$. Clearly $0 \in S^*$ and if $0 \leq s_1 \leq s_2$ and $s_2 \in S^*$, then $s_1 \in S^*$. Next, we will show that $d(s)$ is continuous. Suppose that d is not continuous at s^*. Because $d(s) \in [x_1(t', s), x_2(t', s)]$, we have $s^* \neq 0$. Suppose that $s^k \to s^*$ and $d(s^k) \to z \neq d(s^*)$ as $k \to \infty$. Because $z, d(s^*) \in [x_1(t', s), x_2(t', s)]$ and $x_2(t', s) - x_1(t', s)$ is proportional to $c_3(t')$, we have either $z \gg d(s^*)$ or $d(s^*) \gg z$. If $z \ll y(t', s^*, s_1(t', s^*))$, then there exists $(\bar{s}_k)$ such that if k is large enough, then $p^k \cdot y(t', s^k, \bar{s}_k) \geq p^k \cdot c_t(s^k)$ for some $p^k \in G(y(t', s^k, \bar{s}_k))$ and $\liminf_{k \to \infty} \bar{s}_k = \bar{s} < s_1(t', s^*)$. Taking subsequence, we can assume that $p^k \to p^*$ and $\bar{s}_k \to \bar{s}$ as $k \to \infty$. Now, by definition of $s_1(t', s^*)$, $p \cdot y(t', s^*, \bar{s}) < p \cdot c_t(s^*)$ for any $p \in G(y(t', s^*, \bar{s}))$. Then, by u.h.c. of G, $p^* \in G(y(t', s^*, \bar{s}))$ and thus $p^k \cdot y(t', s^k, \bar{s}_k) < p^k \cdot c_t(s^k)$ for sufficiently large k, a contradiction. Therefore, we have $z \geq y(t', s^*, s_1(t', s^*))$. By the same arguments, we can show that $z \leq y(t', s^*, s_2(t', s^*))$ and thus, $\|c_t(s^*) - d(s^k)\| < r^*$ for sufficiently large k. Choose $s' \in [0, 1]$ such that $G(c_t(s^*) + B_{r^*}) \subset U_{s'}$. Then, by definition of $d(s^k)$, we have $u_{s'}(p) = p \cdot d(s^k)$ for any $p \in G(d(s^k))$.

Now, either $d(s^k) \ll d(s^*)$ or $d(s^k) \gg d(s^*)$ for sufficiently large k. If the former holds, then it implies that $u_{s'}(p) = p \cdot d(s^k) < p \cdot d(s^*)$ for any $p \in G(d(s^k))$. This implies that $u_{s'}(q) = q \cdot d(s^*) > u_{s'}(q)$ for $q \in G(d(s^*))$

[25] Note that lemma 1 still holds under the conditions of theorem 6.

by lemma 10, a contradiction. Similarly, the latter leads a contradiction. Therefore, it does not happen and d is continuous. This automatically implies that S^* is closed.

Choose any $s \in S^*$ and let $G(c_t(s) + B_{r^*}) \subset U_{s'}$. By construction, we have $\|d(s) - c_t(s)\| < r^*$. Because $c_{t'}, d$ are continuous, there exists $\varepsilon'' > 0$ such that $\|d(\bar{s}) - c_t(s)\| < r^*$ and $\|c_{t'}(\bar{s}) - c_t(s)\| < r^*$ for any $\bar{s} \in [s, s + \varepsilon'']$. If $d(\bar{s}) \neq c_{t'}(\bar{s})$ for such $\bar{s}$, then either $d(\bar{s}) \ll c_{t'}(\bar{s})$ or $d(\bar{s}) \gg c_{t'}(\bar{s})$, and we can lead a contradiction by lemma 10. Therefore, we have $s + \varepsilon'' \in S^*$ and S^* is open. This implies that $S^* = [0, 1]$, and thus, if $G(c_t(1) + B_{r^*}) \subset U_{s'}$, then $p \cdot c_{t'}(1) = u_{s'}(p)$ for any $p \in c_{t'}(1)$. Also, $p \cdot c_3(t') = v_t(p)$ for any $p \in G(c_3(t'))$. Note that by lemma 9, $v_{p'} \equiv u_{s'}$ on the intersection of the domain of both function, and $G(c_3(t')) \cup G(c_{t'}(1))$ is included in this intersection if $t' - t$ is sufficiently small. Moreover, $c_{t'}(1)$ is proportional to $c_3(t')$ by definition. Therefore, by lemma 10 we have $c_{t'}(1) = c_3(t')$ for such t', and thus T is open.

It suffices to show that $t^* \equiv \sup T \in T$. By the same arguments as above, we can show that $c_t(s) \in [y(t, s, s_1(t, s)), y(t, s, s_2(t, s))]$ for any $t < t^*$ such that $t^* - t$ is sufficiently small, and $y(t, 1, s_i(t, 1)) \to c_{t^*}(1)$. Therefore, we have $c_3(t) = c_t(1) \to c_{t^*}(1)$ as $t \uparrow t^*$. Meanwhile, we have $c_3(t) \to c_3(t^*)$ as $t \uparrow t^*$ by continuity of the indifference curve. Therefore, $c_{t^*}(1) = c_3(t^*)$ and $t^* \in T$. This completes the proof. ■

Now, choose any $v \in \Omega$. Let $u_v(x) = \frac{\|x\|}{\|v\|}$ if x is proportional to v, and $u_v(x) = \frac{\|c(1)\|}{\|v\|}$ if x is not proportional to v and c is the maximal indifference curve with parameter (x, v). Let

$$x \succsim z \Leftrightarrow u_v(x) \geq u_v(z).$$

If x is proportional to z, then by lemma 15, $x \succsim z$ if and only if $x \geq z$. If x is not proportional to z, then by lemma 16, $x \succsim z$ if and only if $c(1) \geq z$, where c is the maximal indifference curve with parameter (x, z). Lemma 14 says that if $p \in G(x), p \cdot z \leq p \cdot x$ and x is not proportional to z, then $p \cdot c(1) > p \cdot x$ and thus $c(1) \gg z$. Moreover, lemma 15 says that in this case, if $d(t)$ is the maximal indifference curve with parameter (z, x), then $d(1) \ll x$. Therefore, we have $x \succ z$. Clearly, if x is proportional to z and $p \cdot z \leq p \cdot x$, then either $x = z$ or $x \succ z$. Thus, $f^{\succsim}(p, p \cdot x) = x = f(p, p \cdot x)$.

Next, suppose that $f^{\succsim}(p, m) \neq \emptyset$ for some $(p, m) \in \mathbb{R}^n_{++} \setminus A$ with $\sum_i p_i = 1$. Let $x \in f^{\succsim}(p, m)$. If $m > p \cdot x$, then there exists $c > 1$ such that $m \geq p \cdot cx$. Then, $cx \succ x \succ cx$, a contradiction. Hence, we must have $m = p \cdot x$. Choose any $q \in G(x)$ and define $p(t) = (1 - t)q + tp$. Without loss of generality, we can assume that $p(t) \notin G(x)$ if $t > 0$. Then, $(p(t), p(t) \cdot x) \in A$ for sufficiently small $t > 0$. By definition of G, we have $y = f(p(t), p(t) \cdot x) \neq x$. By Walras' law, we have $p(t) \cdot y = p(t) \cdot x$, and thus $p(t) \in G(y)$. By above arguments, we have $x \not\succsim y$. Meanwhile, because $p(t) \cdot y = p(t) \cdot x$, we have either $p \cdot y \le p \cdot x$ or $q \cdot y \le q \cdot x$, and thus $x \succsim y$, a contradiction. Therefore, we have $f^{\succsim}(p, m) = \emptyset$ for any $(p, m) \notin A$. This means that $f^{\succsim} = f$.

Because $\succsim$ has a utility function u_v, it is complete and transitive.

Lemma 17. $\succsim$ is continuous.[26]

Proof. Suppose $x \succ v$. If v is proportional to x, choose a neighborhood U of v such that $z \ll x$ for any $z \in U$. Then, we have $x \succ_r z$, and thus $x \succ z$. If v is not proportional to x, choose the maximal indifference curve $c(t)$ with parameter (x, v), and choose a neighborhood U of v such that $c(1) \gg z$ for any $z \in U$. Then, $c(1) \succ_r z$, and thus $x \sim c(1) \succ z$ for all $z \in U$. Therefore, $\{v | x \succ v\}$ is open.

Next, suppose $v \succ x$. If x is proportional to v, choose a neighborhood U of v such that $z \gg x$ for any $z \in U$. Then, $z \succ_r x$, and thus $z \succ x$. If v is not proportional to x, choose the maximal indifference curve $c(t)$ with parameter (x, v), and choose a neighborhood U of v such that $c(1) \ll z$ for any $z \in U$. Then, we have $z \succ_r c(1)$, and thus $z \succ c(1) \sim x$ for all $z \in U$. Therefore, $\{v | v \succ x\}$ is open.

Now, suppose that $x_k \succsim v_k$ and $(x_k, v_k) \to (x, v) \in \Omega^2$. If $v \succ x$, then there exists $c \in]0, 1[$ such that $cv \succ x$. Clearly $v \succ cv$. Hence, there exists k such that $v_k \succ cv$ and $cv \succ x_k$, implying $v_k \succ x_k$, a contradiction. Therefore, we have $\succsim$ is continuous. ■

Thus, we have (I) implies (II). Next, we shall show the uniqueness of such $\succsim$. Let $f = f^{\succsim'}$ for some complete, transitive, and continuous preference relation $\succsim'$. Because f is surjective, if $y \gneq x$, then $y \succ_r x$, and thus $y \succ' x$. Therefore, $\succsim'$ is strongly monotone. Choose any $x, v \in \Omega$. If x is proportional to v, then $x \succsim v$ if and only if $x \ge v$, if and only if $x \succsim' v$. If x is not proportional to v, let $c(t)$ be the maximal indifference curve with parameter (x, v). We will show that $c(t) \sim' x$ for any $t \in [0, 1]$. Let T be the set of all $\bar{t} \in [0, 1]$ such that if $t \in [0, \bar{t}]$, then $c(t) \sim' x$. We have $0 \in T$ and if $0 \le s \le t$ and $t \in T$, then $s \in T$. Clearly T is a closed subset of $[0, 1]$. Suppose that $t \in T$. By definition, there exists a solution $u_t : U_t \to \mathbb{R}^n_{++}$ of (4) such that U_t is an open and convex neighborhood of $G(c(t))$ and $p \cdot c(s) = u(p)$ for any

[26] In fact, we can show that u_v is continuous. See ch.3 of Mas-Colell, Whinston and Green (1995).

$p \in G(c(s)) \cap U$ with $s \in [0,1]$. Next, define

$$E(p) = \inf\{p \cdot y | y \succsim' x\}.$$

As in the proof of lemma 7, we can show that E is concave, and thus continuous. Moreover, if $(p, E(p)) \in A$, then $DE(p) = f(p, E(p))$. Also, $E(p) = p \cdot c(t)$ if $p \in G(c(t))$. By continuity, there exists an open and convex neighborhood $V \subset U$ of $G(c(t))$ such that $(p, E(p)) \in A$ for any $p \in V$. Then, by lemma 9, we have $E(q) = u(q)$ for any $q \in V$. This implies that there exists $\varepsilon > 0$ such that if $s \in [t, t+\varepsilon[$, then $c(s) = f(p, E(p))$ for any $p \in G(c(s))$, and thus $c(s) \sim' x$. Therefore, T is open and thus $T = [0,1]$.

Hence, $x \succsim' v$ iff either x is proportional to v and $x \geq v$, or x is not proportional to v and $c(1) \geq v$, where $c(t)$ is the maximal indifference curve with parameter (x, v), which implies that $\succsim' = \succsim$.

Now, we will prove (II) implies (I). At first, choose any $(p^*, m^*) \in A$, and let $x = f(p^*, m^*)$. Define $E(p) = \inf\{p \cdot y | y \succsim x\}$. As above, we can show that 1) E is concave and thus continuous, 2) if $(p, E(p)) \in A$, then $DE(p) = f(p, E(p))$, and 3) $E(p^*) = m^*$. By continuity, there exists a neighborhood U of p^* such that $(p, E(p)) \in A$ if $p \in U$. Therefore, by 2), we have $D^2E(p^*) = S_f(p^*, m^*)$, and thus $S_f(p^*, m^*)$ is negative semi-definite and symmetric. Because (p^*, m^*) is arbitrary, we have f satisfies (NSD) and (S).

Next, choose any $x \in \Omega$. Suppose $p, q \in G(x)$ and $r \in [p, q]$. If $r \cdot y \leq r \cdot x$, then either $p \cdot y \leq p \cdot x$ or $q \cdot y \leq q \cdot x$. In both cases, we have $x \succsim y$, and either $x = y$ or $y \not\succsim x$. Therefore, $x = f^{\succsim}(r, r \cdot x) = f(r, r \cdot x)$ and thus $r \in G(x)$. Let (p^k) be a sequence of $G(x)$. Because $\sum_i p_i^k = 1$, it is a bounded sequence, and thus it has a convergent subsequence $(p^{k(\ell)})$. Let $p^{k(\ell)} \to p^*$. At first, suppose that $p^* \notin \mathbb{R}^n_{++}$. Then, there exists i, j such that $p_i^* > 0, p_j^* = 0$. Now, $x + e_j \succ_r x$ and thus $x \not\succsim x + e_j$. By continuity, there exists $\varepsilon > 0$ such that $x \not\succsim x + e_j - \varepsilon e_i$. Because $p^* \cdot (x + e_j - \varepsilon e_i) < p^* \cdot x$, there exists ℓ such that $p^{k(\ell)} \cdot (x + e_j - \varepsilon e_i) < p^{k(\ell)} \cdot x$. Then, $x \neq f(p^{k(\ell)}, p^{k(\ell)} \cdot x)$, a contradiction. Therefore, we have $p^* \in \mathbb{R}^n_{++}$. By continuity of f, we have $x = f(p^*, p^* \cdot x)$, and thus $p^* \in G(x)$. Hence, $G(x)$ is compact.

Suppose $x^k \to x$ as $k \to \infty$, and there exists an open neighborhood U and sequence (p^k) with $p^k \in G(x^k)$ such that $p^k \notin U$ for infinitely many k. Taking subsequence, we can assume that $p^k \notin U$ for any k and $p^k \to p^*$ as $k \to \infty$. Now, by the same arguments as in the proof of theorem 2, we can show that $p^* \in \mathbb{R}^n_{++}$. If $y \in \Omega, p^* \cdot y \leq p^* \cdot x$, then for any $c \in]0,1[$, we have $p^* \cdot (cy) < p^* \cdot x$, and thus for any sufficiently large k, $p^k \cdot (cy) < p^k \cdot x^k$. This implies that $x^k \succsim cy$ and by continuity, $x \succsim cy$. Therefore, again by continuity, $x \succsim y$. Hence, $x = f(p^*, p^* \cdot x)$ and thus $p^* \in G(x)$. Thus, $p^k \in U$

for some k, a contradiction. Therefore, we have G is u.h.c. and f satisfies the C axiom.

Lastly, we should show the NLL axiom holds. Let $t_k \downarrow 0$ and consider $y^k = x + e_j - t_k e_i$. Choose $p^k \in G(y^k)$. Then, by u.h.c. of G, there exists a subsequence $(p^{k(\ell)})$ such that $p^{k(\ell)} \to p^* \in G(x + e_i)$. Because $p^* \cdot (x + e_i) > p^* \cdot x$, we have $p^{k(\ell)} \cdot y^k > p^{k(\ell)} \cdot x$ for some ℓ. This implies that $y^{k(\ell)} \succ_r x$, which completes the proof. ■

6.9 Proof of Theorem 7

For any subset $U \subset \mathscr{F}'$, we say $U \in \mathscr{T}$ if for any $f \in U$, there exists $i \in \mathbb{N}$ and $\varepsilon > 0$ such that $U(f, i, \varepsilon) \subset U$. We will show that $\mathscr{T}$ satisfies the requirements of the topology.

Clearly, $\mathscr{F}'$ and $\emptyset$ is in $\mathscr{T}$, and if $(U_k)_{k \in I}$ is a collection of the elements of $\mathscr{T}$, then $\cup_k U_k$ is also in $\mathscr{T}$. Therefore, it suffices to show that $U_1 \cap U_2 \in \mathscr{T}$ whenever $U_1, U_2 \in \mathscr{T}$. Obviously, it suffices to show that this claim holds for $U_j = U(f_j, i_j, \varepsilon_j), j = 1, 2$. Let $f \in U_1 \cap U_2$, A_j be the domain of f_j, A be the domain of f, and

$$C_j^* = \{(p, m) \in A_j | \|(p, m)\| \in [\frac{1}{i_j}, i_j], \inf_{(q,w) \notin A_j} \|(p, m) - (q, w)\| \geq \frac{1}{i_j}\}, j = 1, 2,$$

$$C_i = \{(p, m) \in A | \|(p, m)\| \in [\frac{1}{i}, i], \inf_{(q,w) \notin A} \|(p, m) - (q, w)\| \geq \frac{1}{i}\}.$$

Then, C_j^* and all C_i are compact, and $C_j^* \subset A$. Because C_i is included in the interior of C_{i+1}, by finite-intersection property, we have $C_1^* \cup C_2^* \subset C_i$ for some i. Then, $U(f, i, \min\{\varepsilon_1, \varepsilon_2\}) \subset U_1 \cap U_2$.

Therefore, we get a topology $\mathscr{T}$ of $\mathscr{F}'$. For some topology $\mathscr{T}'$ of $\mathscr{F}'$, if $U(f, i, \varepsilon)$ is open for any f, i, ε, then $\mathscr{T} \subset \mathscr{T}'$. Thus, $\mathscr{T}$ is the local C^1 topology itself.

Clearly, this topology is Hausdorff. Moreover, the collection $(U(f, i, \frac{1}{i}))_{i \in \mathbb{N}}$ consists of the neighborhood basis of f in topology $\mathscr{T}$, and thus it is first countable.

Next, let (f_k) be a sequence in $\mathscr{F}'$ and $f \in \mathscr{F}'$, and A (resp. A_k) be the domain of f, (resp. f_k,) and

$$C_i = \{(p, m) \in A | \|(p, m)\| \in [\frac{1}{i}, i], \inf_{(q,w) \notin A} \|(q, w) - (p, m)\| \geq \frac{1}{i}\},$$

$$C_i^k = \{(p, m) \in A_k | \|(p, m)\| \in [\frac{1}{i}, i], \inf_{(q,w) \notin A_k} \|(q, w) - (p, m)\| \geq \frac{1}{i}.$$

Suppose that $f_k \to f$ as $k \to \infty$ with respect to $\mathscr{T}$. If $C \subset A$ is a compact set, then $C \subset C_i$ for some i. Then, $f_k \in U(f, i, 1)$ for sufficiently large k. For such k, we have $C \subset A_k$. Moreover, if $\varepsilon > 0$, then $f_k \in U(f, i, \varepsilon)$ for sufficiently large k. This means that $\|f_k - f\|_{C_1} < \varepsilon$ on C if k is sufficiently large. Therefore, $\|f_k - f\|_{C_1} \to 0$ as $k \to \infty$ on C. Conversely, suppose that for any compact set $C \subset A$, $C \subset A_k$ for sufficiently large k and $\|f_k - f\|_{C^1} \to 0$ as $k \to \infty$ on C. To set $C = C_i$ and to choose any $\varepsilon > 0$, we have $f_k \in U(f, i, \varepsilon)$ for sufficiently large k. This implies that $f_k \to f$ as $k \to \infty$ with respect to $\mathscr{T}$. This completes the proof. ■

6.10 Proof of Theorem 8

At first, suppose that $(f_k)_k$ converges to f with respect to the local C^1 topology. Let $\succsim_k = H(f_k)$ and $\succsim = H(f)$. Choose any $x, v \in \Omega$ such that x is not proportional to v. Let $c(t)$ (resp. $c_k(t)$) be the maximal indifference curve corresponding to f (resp. f_k) with parameter (x, v). We will show that $c_k(1)$ converves to $c(1)$. To show this, let T be the set of all $\bar{t}$ such that $c_k(t) \to c(t)$ for any $t \in [0, \bar{t}]$. Clearly $0 \in T$.

Next, if $0 \le s < t \le 1$, then $c(t) \in [x_1^*(t), x_2^*(t)]$, where $x_i^*(t) = \frac{1-t}{1-s}c(s) + \frac{t-s}{1-s}y_i(c(s), v)$. Therefore,

$$\begin{aligned}\|c(s) - c(t)\| &\le \max_{i=1,2} \|c(s) - x_i^*(t)\| \\ &\quad \max_i \frac{t-s}{1-s} \|y_i(c(s), v) - c(s)\| \\ &\le L(t-s),\end{aligned}$$

where $L = \max_{i=1,2} \|y_i(x, v) - x\|$.[27] By the same arguments, we have

$$\|c_k(s) - c_k(t)\| \le L(t-s).$$

Therefore, we get the same Lipschitz constant L of c, c_k. Let $t^* = \sup T$ and choose any ε. Then, there exists $t \in T$ such that $t^* - t < \frac{\varepsilon}{3L}$. If k is sufficiently large, then $\|c(t) - c_k(t)\| < \frac{\varepsilon}{3}$. Then,

$$\|c(t^*) - c_k(t^*)\| \le \|c(t^*) - c(t)\| + \|c(t) - c_k(t)\| + \|c_k(t) - c_k(t^*)\| < \varepsilon,$$

and thus $t^* \in T$ and T is closed.

Therefore, to show $T = [0, 1]$, it suffices to show that T is open. Because $c((1-s)t + s)$ (resp. $c_k((1-s)t + s)$) is the maximal indifference curves corresponding to f (resp. f_k) with parameter $(c(t), v)$, (resp. $(c_k(t), v)$,) it suffices to show that T includes the neighborhood of 0.

[27] Because $v_i(x, v) = v_i(c(s), v)$, we can show that $y_i(c(s), v) - c(s) = (1-s)y_i(x, v) - x$.

Let G be the inverse demand correspondence and $C = \{(q,w)|\exists p \in G(x), \|(q,w)-(p,p\cdot x)\| \le \varepsilon\}$. If $\varepsilon > 0$ is sufficiently small, then C is included in the domain of f. By lemma 12, there exists a solution $u : U \to \mathbb{R}_{++}$ of

$$Du(q) = f(q, u(q)),$$

such that U is an open and convex neighborhood of $G(x)$, $u(p) = p \cdot x$ for any $p \in G(x)$, and $\{(q,u(q))|q \in U\}$ is included in the interior of C.

Now, choose any $p \in G(x)$, and consider the following equation:

$$\dot{d}(t) = I(t, d(t); p, q, p', m, F), d(0) = p \cdot x, \tag{5}$$

where F is defined on C and C^1-class, and $I(t,d;p,q,p',m,F) = F((1-t)p' + tq, d + m - p \cdot x)$. If $d^*(t;p,q) = u((1-t)p + tq)$, then $d^*(t;p,q)$ is a solution of this equation with $p' = p, m = p \cdot x$ and $F = f$. Choose any $\varepsilon' > 0$. Because $((1-t)p + tq, d^*(t;p,q))$ belongs to the interior of C, there exists $a > 0$ such that if $\inf_{t\in[0,1],q\in U} \|(q',m') - ((1-t)p+tq, d^*(t;p,q))\| \le a$, then $(q',m') \in C$. Choose any $b > 0$ and let

$$\Pi(p,q,b) = \{(d,p',m,F)|\max\{\|p'-p\|, |m-p\cdot x|, \|F-f\|_{C^1}\} < b\},$$

where $\|\cdot\|_{C^1}$ is the C^1 norm on C. If a, b is sufficiently small, then for any $p \in G(x), q \in U, \inf_{t\in[0,1]} |d - d^*(t;p,q)| < a$ and $(p',m,F) \in \Pi(p,q,b)$, (p',m) is in the interior of C, and $((1-t)p' + tq, d + m - p \cdot x) \in C$ for any $t \in [0,1]$. Therefore, there exists $L_1, L_2 > 0$ such that $\|\frac{\partial F}{\partial d}((1-t)p'+tq, d+m-p\cdot x)\| \le L_1$, and

$$|I(t,d;p,q,p',m,F) - I(t,d;p,q,p,p\cdot x,f)| \le L_2 \max\{\|p'-p\|, |m-p\cdot x|, \|F-f\|_{C^1}\},$$

for any $t \in [0,1]$. Let $(p',m,F) \in \Pi(p,q,b)$, and $t_{p,q,p',m,F} > 0$ be the supremum of $\bar{t} > 0$ such that the solution $d(t;p,q,p',m,F)$ of (5) can be defined on $[0,\bar{t}]$. By the same arguments as in the proof of theorem 4, we can show that

$$|d(t;p,q,p',m,F) - d^*(t;p,q)| \le M \max\{\|p'-p\|, |m-p\cdot x|, \|F-f\|_{C^1}\}$$

for any $t \in [0, t_{p,q,p',m,F}]$, where $M > 0$ be some constant independent to p, q. If $b' > 0$ and $Mb' < a$, then for any $(p',m,F) \in \Pi(p,q,b')$, $t_{p,q,p',m,F} = 1$ and

$$|d(t;p,q,p',m,F) - d^*(t;p,q)| \le M \max\{\|p'-p\|, |m-p\cdot x|, \|F-f\|_{C^1}\} < a,$$

for any $t \in [0,1]$. Note that $d^*(1;p,q) = u(q)$ and thus it is independnt to the choice of p. Also, we have $((1-t)p' + tq, d(t;p,q,p',m,F)) \in C$.

Now, define $x_i(t) = (1-t)x + ty_i(x,v)$. As in the proof of lemma 14, we can show that there exists $\bar{t} > 0$ such that if $t \in [0,\bar{t}]$, then $\cup_{y\in[x_1(t),x_2(t)]} G(y) \subset U$, and $c(t)$ is the unique $y \in [x_1(t), x_2(t)]$ such that there exists $q \in G(y)$ with $y = f(q, u(q))$.[28] Choose such q.

Note that $(c_k(t))$ is a sequence on $[x_1(t), x_2(t)]$, and thus if y is a limit point of this sequence, then either $y \geq c(t)$ or $y \leq c(t)$. Suppose that $(c_k(t))$ has a limit point y such that $y \gg c(t)$. Let $z = \frac{y+c(t)}{2}$ $p \in G(x)$, and define $x_k = f_k(p, p\cdot x)$. For any sufficiently large k, the domain of f_k includes C and $\|f_k - f\|_{C^1} < b'$. Therefore, the solution $d_k(t) = d(t;p,q,p,p\cdot x, f_k)$ exists and $|d_k(1) - d^*(1;p,q)| \to 0$ as $k \to \infty$. Now, let $E_k(r) = \inf\{r\cdot y | y \succsim_k x_k\}$ and $S_k = \{\bar{t} \in [0,1] | \forall t \in [0,\bar{t}], d_k(t) = E_k((1-t)p + tq)\}$. Clearly $E_k(p) = p\cdot x = d_k(0)$ and $0 \in S_k$. Also, because E_k is concave, it is continuous and S_k is closed. Now, let $\bar{t} \in S_k$. Then, $((1-t)p + tq, E_k((1-t)p+tq)) \in A$ for any $t \in [0,\bar{t}]$, and thus,

$$DE_k(r) = f(r, E_k(r))$$

for any r such that $\inf_{t\in[0,\bar{t}]} \|r - [(1-t)p+tq]\|$ is sufficiently small. Therefore, $s \mapsto E_k((1-s)p + sq)$ satisfies (5) with $(p,q,p,p\cdot x, f_k)$, and thus it is the same as $d_k(s)$ on some neighborhood of t, which implies that S_k is open. Therefore, $S_k = [0,1]$ and we have $f_k(q, d_k(1)) \sim_k x_k$. Note that $p\cdot x_k = p\cdot x$ and thus $x_k \succsim_k x$. Meanwhile, because y is a limit point of $(c_k(t))$, there exist infinitely many k such that $c_k(t) \gg z$. For such k, $x \sim_k c_k(t) \succ_k z$. However, because $f_k(q, d_k(1)) \to f(q, d^*(1;p,q)) = c(t)$, we have $z \succ_k x_k$ for sufficiently large k, a contradiction. Therefore, $(c_k(t))$ does not have such a limit point.

Next, suppose that $(c_k(t))$ has a limit point y such that $y \ll c(t)$. Let $z = \frac{y+c(t)}{2}$. Choose $\varepsilon' > 0$ so small that if $\|c(t) - w\| < \varepsilon'$, then $w \gg z$. If k is sufficiently large, then C is included in the domain of f_k, and $\|f_k - f\|_{C^1} < \min\{b', \frac{\varepsilon'}{M}\}$. Moreover, there exists $\delta \in]0,1[, p \in G(x)$ and $p' \in G(\delta x)$ such that $\|p - p'\|, |p'\cdot\delta x - p\cdot x| < \min\{b', \frac{\varepsilon'}{M}\}$. Then, we can define $d^k(t) = d(t;p,q,p',p'\cdot\delta x, f_k)$ for any sufficiently large k. Define $z_k = f_k(p', d^k(0))$ and $v_k = f_k(q, d^k(1))$. By the same arguments as above, we have $z_k \sim_k v_k$. Because $z_k \to \delta x$ and $x \gg \delta x$, we have $x \succ_k z_k$ for sufficiently large k. Meanwhile, we have $\|v_k - c(t)\| < \varepsilon'$, and thus $v_k \gg z$. Therefore, $z_k \succ_k z$. Finally, because y is a limit point of $(c_k(t))$, there exist infinitely many k such that $z \gg c_k(t)$, and thus $z \succ_k c_k(t) \sim_k x$, a contradiction.

Therefore, we have $c_k(t) \to c(t)$ as $k \to \infty$. This implies that T includes $[0,\bar{t}]$, and thus $T = [0,1]$.

[28] Use the uniqueness of the indifference curve.

Now, let $(x, v) \in \succsim$ and choose any neighborhood U of (x, v). Then, it includes (x, dv) for some $d \in]0, 1[$. By the above arguments, $(x, dv) \in \succsim_k$ for sufficiently large k, and thus $(x, v) \in \liminf_k \succsim_k$.

Next, let $(x, v) \notin \succsim$. Let $v^* = c_1 v, x^* = c_2 x, c_2 > 1 > c_1$ and $v^* \succ x^*$. Then, there exists an open neighborhood U of (x, v) such that for any $(y, z) \in U$, $y \ll x^*$ and $z \gg v^*$. By above arguments, we have $v^* \succ_k x^*$ for sufficiently large k. For such k, we have

$$z \succ_k v^* \succ_k x^* \succ_k y,$$

for any $(y, z) \in U$, and thus $(x, v) \notin \limsup_k \succsim_k$. This implies that $\succsim = \lim_k \succsim_k$. This completes the proof. ■

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