# Cosine similarity and the Borda rule*

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**Abstract**

Cosine similarity is a commonly used similarity measure in computer science. We propose a voting rule based on cosine similarity, namely, the *cosine similarity rule*. The proposed voting rule selects a social ranking that maximizes cosine similarity between the social ranking and a given preference profile. We show that the cosine similarity rule in fact coincides with the Borda rule. We also discuss an analogous relation between the Borda rule and the Condorcet rule from viewpoints of distance and similarity.

*Keywords:* The Borda rule, Cosine similarity.

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# 1 Introduction

*Cosine similarity* is a commonly used similarity measure in computer science. It has a variety of applications such as data mining, text mining, and information retrieval.[1] We apply this similarity measure to define a new voting rule in social choice theory, namely, the *cosine similarity rule*. The rule selects a social ranking that maximizes cosine similarity between the social ranking and a given preference profile. Our main finding is that the cosine similarity rule in fact coincides with the Borda rule.

The Borda rule is one of the most important voting rules in social choice theory, which is introduced by Jean-Charles de Borda (1784). It is known that Borda's choice method has many desirable properties such as maximization of the average share of votes in pairwise comparison (Black 1976, Coughlin 1979), closest proximity to unanimous agreement (Farkas and Nitzan 1979; Sen 1977), avoidance of many paradoxes observed in positional rules (Saari 1989), and avoidance of the Condorcet loser (Fishburn and Gehrlein 1976, Okamoto and Sakai 2013). In particular, Young (1974) characterizes Borda's choice method by a set of desirable properties: neutrality, consistency, faithfulness and cancellation property. On the other hand, there are a few studies on Borda's *ranking* method. In this paper, we focus on the Borda rule as a ranking method. We provide a rationale for the use of the Borda rule from a new viewpoint. It selects the ranking closest to a given preference profile when measured by cosine similarity. More formally, for each preference profile, the Borda ranking maximizes the sum of cosine similarities between itself and each voter's preference.

Kemeny (1959) introduces this type of approach to search for desirable ranking methods based on similarity or distance from a given preference profile. Kemeny (1959) defines a metric that measures the distance between two rankings, so called *Kemeny distance*[2]. Then he proposes the ranking method that selects a ranking minimizing the sum of Kemeny distances between the social ranking and each voter's preference. Surprisingly, it is the unique ranking method that satisfies neutrality, consistency, and the Condorcet criterion (Young and Levenglick 1978). Moreover, Young

[1] See, for example, Tan, Steinbach and Kumar (2005).

[2] The main part of his analysis is the axiomatization of Kemeny distance. He characterizes the distance by the three axioms of metrics, neutrality, local independence, betweenness, and normalization (Kemeny 1959; Kemeny and Snell 1962).

(1988) finds that what Condorcet (1785) had in mind was in fact the maximum likelihood method, and he shows that Kemeny's rule coincides with the maximum likelihood method. In short, Condorcet's rule selects a ranking that minimizes the sum of Kemeny distances from voter's preferences.

"Borda or Condorcet, which is better?" is the two century old question in social choice theory[3]. In the context, it is natural to check if the Borda ranking also minimizes some distance or maximize some similarity from voter's preferences. If we know that, we can compare the Borda rule with the Condorcet rule from the same viewpoints of distance and similarity. In fact, our result ensures that the Borda rule selects a ranking that maximizes the sum of cosine similarities from voter's preferences. This result is analogous to the main result of Young and Levenglick (1978).

This paper is organized as follows. In Section 2 we introduce definitions. In Section 3 we show the equivalence between the Borda rule and the cosine similarity rule. In Section 4 we restrict the range of ranking methods to linear orderings and show another equivalence theorem. Section 5 concludes our discussion.

## 2 Definitions

Let $I = \{1, 2, \dots, n\}$ be the finite set of voters and $A = \{a_1, a_2, \dots, a_m\}$ the finite set of alternatives. Let $\mathscr{R} \subset A \times A$ be the set of *complete* and *transitive* binary relations on $A$ and $\mathscr{P} \subset A \times A$ the set of *complete*, *transitive* and *anti-symmetric* binary relations on $A$.[4] Each voter $i \in I$ has a preference $\succsim_i \in \mathscr{P}$ on $A$. A *preference profile* is a list of preferences

$$\succsim \equiv (\succsim_i)_{i \in I} \in \mathscr{P}^n.$$

A *ranking method* is a function $F$ that maps each preference profile $\succsim \in \mathscr{P}^n$ to a social ranking $F(\succsim) \in \mathscr{R}$.

For each voter $i \in I$, each alternative $a \in A$ and each preference $\succsim_i \in \mathscr{P}$, let

$$r_a(\succsim_i) \equiv |\{a' \in A : a \succsim_i a'\}|$$

[3] See, for example, Saari (2006).

[4] A binary relation $\succsim$ is *complete* if for any $a, b \in A$, $a \succsim b$ or $b \succsim a$. It is *transitive* if for any $a, b, c \in A$, $[a \succsim b$ and $b \succsim c]$ implies $a \succsim c$. It is *anti-symmetric* if for any $a, b \in A$, $[a \succsim b$ and $b \succsim a]$ implies $a = b$.

be the *rank* of $a$ in $\succsim_i$. For each $\succsim_i \in \mathscr{P}$, define

$$r(\succsim_i) \equiv (r_{a_1}(\succsim_i), r_{a_2}(\succsim_i), \ldots, r_{a_m}(\succsim_i)) \in \mathbb{N}^m.$$

We call $r(\succsim_i)$ the *rank expression* of $\succsim_i$. A preference $\succsim_i$ and its rank expression $r(\succsim_i)$ have the same information. The *Borda score* of $a$ in $\succsim$ is given by

$$S(a, \succsim) \equiv \sum_{i \in I} r_a(\succsim_i).$$

The following ranking method is proposed by Borda (1784).

**Definition 1** (Borda ranking method)**.** The *Borda ranking method* is the ranking method $F^B$ such that for each $\succsim \in \mathscr{P}^n$ and each $a, b \in A$,

$$a \, F^B(\succsim) \, b \text{ if and only if } S(a, \succsim) \geq S(b, \succsim).$$

Next, we define a similarity measure that plays a key role in our analysis.

**Definition 2** (Cosine similarity)**.** For each vector $x, y \in \mathbb{R}^m_{++}$, the *cosine similarity* between $x$ and $y$ is

$$C(x,y) \equiv \frac{x \cdot y}{||x|| \, ||y||},$$

where $||x||$ is the Euclidean norm of $x$, and $x \cdot y$ denotes the inner product between $x$ and $y$.

The *cosine similarity* is a commonly used similarity measure between two vectors. By definition, $C(x,y) = \cos\theta_{x,y}$, where $\theta_{x,y}$ is the angle between $x$ and $y$. Therefore, $C(x,x) = \cos 0 = 1$ and $C(x,y) \leq 1$ for all $x, y \in \mathbb{R}^m_{++}$. If two vectors $x$ and $y$ are similar, the value of $C(x,y)$ is close to 1.

For example, consider the following four vectors $x, y, z$ and $w$.

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}, \quad z = \begin{pmatrix} 10 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}.$$

We have

$$
\begin{aligned}
C(x,y) &= \frac{1\cdot 1+2\cdot 2+3\cdot 3+4\cdot 2}{\sqrt{1^2+2^2+3^2+4^2}\sqrt{1^2+2^2+3^2+2^2}} \fallingdotseq .9467,\\
C(x,z) &= \frac{1\cdot 10+2\cdot 1+3\cdot 1+4\cdot 1}{\sqrt{1^2+2^2+3^2+4^2}\sqrt{10^2+1^2+1^2+1^2}} \fallingdotseq .3418,\\
C(x,w) &= \frac{1\cdot 2+2\cdot 4+3\cdot 6+4\cdot 8}{\sqrt{1^2+2^2+3^2+4^2}\sqrt{2^2+4^2+6^2+8^2}} = 1.
\end{aligned}
$$

Note that cosine similarity measures the similarity of orientations of two vectors and is independent from their lengths. Using cosine similarity and rank expressions of preferences, we can calculate the similarity between two preferences.

**Definition 3** (Cosine similarity between two preferences)**.** For each $\succsim_i$ and $\succsim_j \in \mathscr{P}$, the *cosine similarity* between $\succsim_i$ and $\succsim_j$ is $C(r(\succsim_i), r(\succsim_j))$.

A social ranking can be a weak ordering that cannot be associated with rank expression $r$, so we introduce the following notation. For each vector $x \in \mathbb{R}^m_{++}$, we call $R(x)$ the *ranking expressed by* $x$ if for each $k, \ell \in \{1, 2, \ldots, m\}$,

$$a_k \, R(x) \, a_\ell \Longleftrightarrow x_k \geq x_\ell.$$

The following ranking method is proposed in this paper.

**Definition 4** (Cosine similarity ranking method)**.** The *cosine similarity ranking method* is the ranking method $F^C$ such that for each $\succsim \in \mathscr{P}^n$,

$$F^C(\succsim) = R(x), \text{ where } x \text{ maximizes } \sum_{i\in I} C(x, r(\succsim_i)).$$

This rule selects a social ranking that maximizes the sum of cosine similarities between the social ranking and each voter's ranking[5].

[5] In the definition, we use vector expression $x$. That point is different from Young and Levenglick (1978) but it is not essential for our results. In Section 4, we consider only linear orderings same as Young and Levenglick (1978) and define another cosine similarity ranking method without using vector expression $x$.

# 3 Equivalence theorem

We are now in a position to state our main result. We show that the cosine similarity ranking coincides with the Borda ranking.

**Theorem 1.** *For each* $\succsim \in \mathscr{P}^n$,

$$F^C(\succsim) = F^B(\succsim).$$

In other words, for each preference profile, the Borda rule selects the social ranking closest to the preference profile when measured by cosine similarity. As before mentioned, the Kemeny rule, which selects the social ranking closest to a given preference profile when measured by Kemeny distance, coincides with the Condorcet rule. Figure 1 illustrates the relations between the Kemeny rule, the Condorcet rule, the cosine similarity rule, and the Borda rule. While Condorcet's rule minimizes Kemeny distance, Borda's rule maximizes cosine similarity.

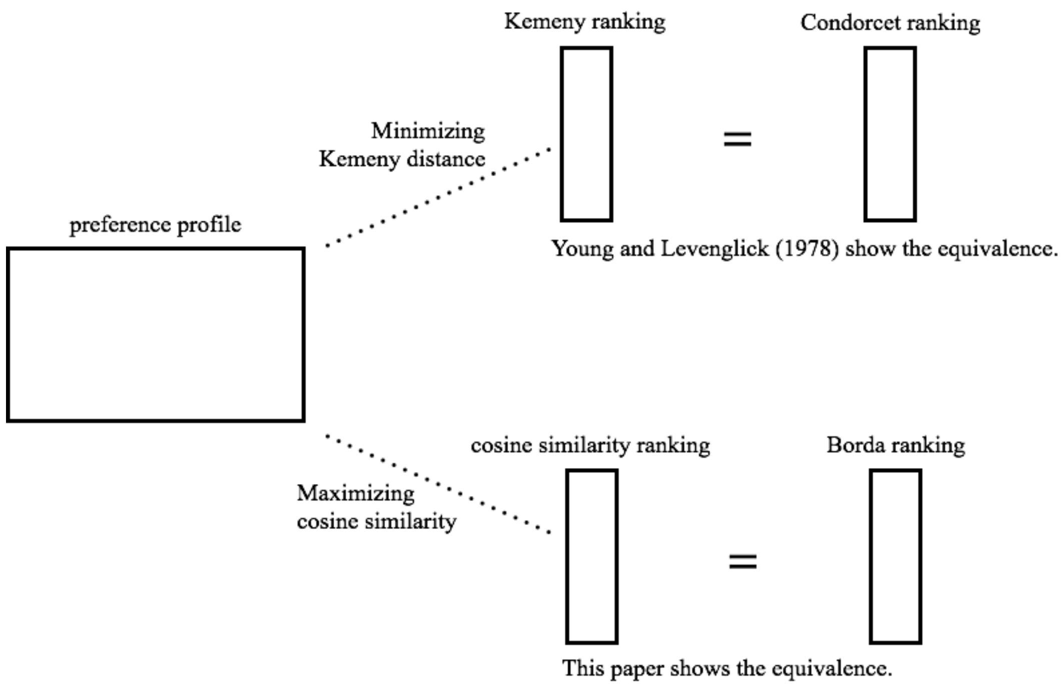


Figure 1: Borda's rule and Condorcet's rule

*Proof.* Take any $\succsim \in \mathscr{P}^n$. We show that if $x \in \mathbb{N}^m$ maximizes

$$\sum_{i \in I} C(x, r(\succsim_i)),$$

then there exists $\alpha \in \mathbb{R}_{++}$ such that $x_k = \alpha \cdot S(a_k, \succsim)$ for all $k = 1, 2, \ldots, m$.

Assume that $x$ maximizes $\sum_{i \in I} C(x, r(\succsim_i))$. By the definition of cosine similarity,

$$\begin{aligned}\sum_{i \in I} C(x, r(\succsim_i)) &= \sum_{i \in I} \frac{x \cdot r(\succsim_i)}{\|x\| \|r(\succsim_i)\|} \\ &= \sum_{i \in I} \frac{\sum_{j=1}^m x_j r_{a_j}(\succsim_i)}{\|x\| \|r(\succsim_i)\|}.\end{aligned}$$

Let $\|r\| \equiv \|r(\succsim_i)\| = \sqrt{1^2 + 2^2 + \cdots + m^2}$ for all $i \in I$. By the commutative property of addition,

$$\begin{aligned}\sum_{i \in I} \frac{\sum_{j=1}^m x_j r_{a_j}(\succsim_i)}{\|x\| \|r(\succsim_i)\|} &= \frac{1}{\|r\|} \sum_{i \in I} \frac{\sum_{j=1}^m x_j r_{a_j}(\succsim_i)}{\|x\|} \\ &= \frac{1}{\|r\|} \sum_{j=1}^m \frac{x_j \sum_{i \in I} r_{a_j}(\succsim_i)}{\|x\|}.\end{aligned}$$

Since $\sum_{i \in I}^m r_{a_j}(\succsim_i) = S(a_j, \succsim)$,

$$\begin{aligned}\frac{1}{\|r\|} \sum_{j=1}^m \frac{x_j \sum_{i \in I} r_{a_j}(\succsim_i)}{\|x\|} &= \frac{1}{\|r\|} \sum_{j=1}^m \frac{x_j S(a_j, \succsim)}{\|x\|} \\ &= \frac{\|(S(a_j, \succsim))_{j=1}^m\|}{\|r\|} \frac{\sum_{j=1}^m x_j S(a_j, \succsim)}{\|x\| \|(S(a_j, \succsim))_{j=1}^m\|} \\ &= \frac{\|(S(a_j, \succsim))_{j=1}^m\|}{\|r\|} \frac{x \cdot (S(a_j, \succsim))_{j=1}^m}{\|x\| \|(S(a_j, \succsim))_{j=1}^m\|}.\end{aligned}$$

By the definition of cosine similarity,

$$\frac{x \cdot (S(a_j, \succsim))_{j=1}^m}{\|x\| \|(S(a_j, \succsim))_{j=1}^m\|} = C(x, S(a_j, \succsim))_{j=1}^m) \leq 1.$$

Therefore,

$$\sum_{i\in I} C(x, r(\succsim_i)) \leq \frac{\|(S(a_j,\succsim))_{j=1}^m\|}{\|r\|}. \tag{1}$$

Since $C((S(a_j,\succsim))_{j=1}^m, (S(a_j,\succsim))_{j=1}^m) = 1$, by *scale invariance property* of cosine similarity,[6]

$$\sum_{i\in I} C(x, r(\succsim_i)) = \frac{\|(S(a_j,\succsim))_{j=1}^m\|}{\|r\|} \Longleftrightarrow x = \alpha \cdot (S(a_j,\succsim))_{j=1}^m,$$

where $\alpha$ is a positive real number. Thus, if $x$ maximizes the left hand side of (1), then there exists $\alpha > 0$ such that $x_k = \alpha \cdot S(a_k,\succsim)$ for all $k = 1,2,\ldots,m$. Assume that $x$ maximizes the left hand side of (1). By the definition of $R(x)$,

$$x_k \geq x_j \Longleftrightarrow S(a_k,\succsim) \geq S(a_j,\succsim) \Longleftrightarrow a_k\, R(x)\, a_j. \tag{2}$$

Therefore, the ranking $R(x)$ is uniquely determined by $\succsim$, which implies that $F^C(\succsim) = R(x)$ is uniquely determined by $\succsim$. In addition, by (2) and the definition of the Borda ranking, $F^B(\succsim) = R(x)$. Therefore, $F^C(\succsim) = F^B(\succsim)$. □

# 4 Linear orderings and another equivalence

Even though voter's preferences are linear orderings, there may be cases in which two alternatives get the same scores. Therefore, it is natural to assume that the range of ranking methods is the set of weak orderings. Indeed, in the previous section, we establish the equivalence between the weak ordering Borda rankings and the weak ordering cosine similarity rankings. However, Young and Levenglick (1978) show the equivalence between the Kemeny ranking and the Condorcet rankings on the restricted range of linear orderings. By considering only linear orderings, we can clarify analogous relation between Young's and Levenglick's (1978) analysis and our analysis In this section, we show the equivalence between the linear ordering Borda rankings and

[6] Cosine similarity is scale invariant, that is, for each $x,y \in \mathbb{R}^m_{++}$ and for each $\alpha > 0$, $C(\alpha \cdot x, y) = C(x,y)$. Other properties of cosine similarity are discussed in Section 5.

the linear ordering cosine similarity rankings.

**Definition 5** (Linear Borda ranking method)**.** The *linear Borda ranking method* is a correspondence $F^{LB} : \mathscr{P}^n \twoheadrightarrow \mathscr{P}$ such that for each $\succsim \in \mathscr{P}^n$,

$$F^{LB}(\succsim) = \{P \in \mathscr{P} : a_j \, P \, a_k \Longrightarrow S(a_j, \succsim) \geq S(a_k, \succsim) \; \forall a_j, a_k \in A\}. \tag{3}$$

For example, suppose $S(a,\succsim) = 11, S(b,\succsim) = 14, S(c,\succsim) = 11$, and $S(d,\succsim) = 4$. Then the set of linear Borda rankings is

$$F^{LB}(\succsim) = \{P, P'\},$$

where

$$b \, P \, c \, P \, a \, P \, d \quad \text{and} \quad b \, P' \, a \, P' \, c \, P' \, d.$$

**Definition 6** (Linear cosine similarity ranking method)**.** The *linear cosine similarity ranking method* is a correspondence $F^{LC} : \mathscr{P}^n \twoheadrightarrow \mathscr{P}$ such that for each $\succsim \in \mathscr{P}^n$,

$$F^{LC}(\succsim) = \{P \in \mathscr{P} : P \text{ maximizes } \sum_{i \in I} C(r(P), r(\succsim_i))\}.$$

The linear cosine similarity rule can be defined without using vector expressions, so its definition is simpler than that of the weak ordering cosine similarity rule.

**Theorem 2.** *For each* $\succsim \in \mathscr{P}^n$*,*

$$F^{LB}(\succsim) = F^{LC}(\succsim).$$

Before giving the proof, we offer a lemma.

**Lemma 1.** *For any* $x_1 \leq x_2 \leq \cdots \leq x_m$*,* $y_1 \leq y_2 \leq \cdots \leq y_m$ *and any permutation* $\sigma$ *on* $\{1,2,\ldots,m\}$*,*

$$\sum_{k=1}^{m} x_k y_k \geq \sum_{k=1}^{m} x_k y_{\sigma(k)}.$$

*Proof.* Consider any $x, y \in \mathbb{R}^m$ with $x_1 \leq x_2 \leq \cdots \leq x_m$ and $y_1 \leq y_2 \leq \cdots \leq y_m$. Consider any permutation $\sigma$ on $\{1,2,\ldots,m\}$. We prove this lemma by mathematical induction.

First, we shall show that if $\sigma(m) \neq m$, then there exists a permutation $\pi$ on $\{1, 2, \ldots, m\}$ such that

$$\sum_{k=1}^{m} x_k y_{\pi(k)} \geq \sum_{k=1}^{m} x_k y_{\sigma(k)}. \tag{4}$$

Assume that $\sigma(m) \neq m$. Then, there exist $i, j < m$ such that $\sigma(m) = i$ and $\sigma(j) = m$. Since $x_m \geq x_j$ and $y_m \geq y_i$, $(x_m - x_j)(y_m - y_i) \geq 0$. Therefore,

$$x_m y_m + x_j y_i \geq x_m y_i + x_j y_m. \tag{5}$$

Let $\pi$ be the permutation on $\{1, 2, \ldots, m\}$ such that

$$\pi(k) = \begin{cases} m & \text{if } \; k = m, \\ i & \text{if } \; k = j, \\ \sigma(k) & \text{if } \; k \notin \{m, j\}. \end{cases}$$

By (5),

$$\begin{aligned} \sum_{k=1}^{m} x_k y_{\pi(k)} &= \sum_{k \neq m, j} x_k y_{\sigma(k)} + x_m y_m + x_j y_i \\ &\geq \sum_{k \neq m, j} x_k y_{\sigma(k)} + x_m y_i + x_j y_m \\ &\geq \sum_{k=1}^{m} x_k y_{\sigma(k)}, \end{aligned}$$

that is, we have (4).

Next, we show that if $\sigma(m) = m$ and $\sigma(m-1) \neq m-1$, then there exists a permutation $\pi$ such that

$$\sum_{k=1}^{m} x_k y_{\pi(k)} \geq \sum_{k=1}^{m} x_k y_{\sigma(k)}. \tag{6}$$

Assume that $\sigma(m) = m$ and $\sigma(m-1) \neq m-1$. Then, there exist $i, j < m-1$ such that $\sigma(m-1) = i$ and $\sigma(j) = m-1$. Since $x_{m-1} \geq x_j$ and $y_{m-1} \geq y_i$, we have $(x_{m-1} - x_j)(y_{m-1} - y_i) \geq 0$. Therefore,

$$x_{m-1} y_{m-1} + x_j y_i \geq x_{m-1} y_i + x_j y_{m-1}. \tag{7}$$

Let $\pi$ be such that

$$\pi(k) = \begin{cases} m-1 & \text{if} \quad k = m-1, \\ i & \text{if} \quad k = j, \\ \sigma(k) & \text{if} \quad k \notin \{m-1, j\}. \end{cases}$$

By (7),

$$\begin{aligned} \sum_{k=1}^{m} x_k y_{\pi(k)} &= \sum_{k \neq m-1, j} x_k y_{\sigma(k)} + x_{m-1} y_{m-1} + x_j y_i \\ &\geq \sum_{k \neq m-1, j} x_k y_{\sigma(k)} + x_{m-1} y_i + x_j y_{m-1} \\ &\geq \sum_{k=1}^{m} x_k y_{\sigma(k)}, \end{aligned}$$

that is, we have (6).

In general, take any $\ell \in \{1, 2, \ldots, m\}$ and assume that $\sigma(m) = m, \sigma(m-1) = m-1, \ldots, \sigma(\ell+1) = \ell+1$, and $\sigma(\ell) \neq \ell$. Let $\pi$ be identical to $\sigma$ except that $\pi(\ell) = \ell$ and $\pi(\sigma^{-1}(\ell)) = \sigma(\ell)$. Then,

$$\sum_{k=1}^{m} x_k y_{\pi(k)} \geq \sum_{k=1}^{m} x_k y_{\sigma(k)}.$$

Therefore, $\sum_{k=1}^{m} x_k y_k \geq \sum_{k=1}^{m} x_k y_{\sigma(k)}$. □

*Proof of Theorem 2.* Consider any $\succsim \in \mathscr{P}^n$. Let us show $F^{LC}(\succsim) \subset F^{LB}(\succsim)$. Take any $P \in F^{LC}(\succsim)$. Let $s \in \mathbb{N}^m$ be such that $s_j = S(a_j, \succsim)$ for each $a_j \in A$. By an argument similar to the proof of Theorem 1,

$$\begin{aligned} \sum_{i \in I} C(r(P), r(\succsim_i)) &= \frac{\|s\|}{\|r\|} \frac{r(P) \cdot s}{\|r(P)\| \|s\|} \\ &= \frac{r_1(P)s_1 + r_2(P)s_2 + \cdots + r_m(P)s_m}{\|r\| \|r(P)\|}. \end{aligned}$$

Since $P$ is a linear order, $\|r(P)\| = \sqrt{\frac{1}{6} m(m+1)(2m+1)}$, which is independent of the choice of $P$. Thus, $P$ maximizes $\sum_{i \in I} C(r(P), r(\succsim_i))$ if and only if $P$ maximizes

$r_1(P)s_1 + r_2(P)s_2 + \cdots + r_m(P)s_m$. Since $P \in F^{LC}(\succsim)$, $P$ maximizes $\sum_{i \in I} C(r(P), r(\succsim_i))$. By Lemma 1,

$$r_j(P) > r_k(P) \Longrightarrow s_j \geq s_k \tag{8}$$

for each $j,k = 1,2,\ldots,m$. Since $r_j(P) > r_k(P)$ implies $a_j \, P \, a_k$, we have (3). Therefore $P \in F^{LB}(\succsim)$, which in turn implies $F^{LC}(\succsim) \subset F^{LB}(\succsim)$. Proving $F^{LB}(\succsim) \subset F^{LC}(\succsim)$ is similar because if $P \in F^{LB}(\succsim)$, then $P$ satisfies (8). It implies that $P$ maximizes $\sum_{i \in I} C(r(P), r(\succsim_i))$ by Lemma 1. □

# 5 Conclusion

We have proposed a new voting rule, the cosine similarity rule. We have shown that the cosine similarity rule coincides with the Borda rule. Our analysis provides a rationale for the use of the Borda rule from the viewpoint of similarity. Because the Borda rule selects a social ranking that is closest to voters' preferences, we can say that it is suitable for democratic decision makings. Moreover, we have found an analogous relation between Borda's rule and Condorcet's rule. While the consistently-extended Condorcet rule coincides with the Kemeny rule, the Borda rule coincides with the cosine similarity rule. By our analysis, we can compare them from the same perspectives of distance and similarity.

Finally, we examine properties of the cosine similarity operator $C$. Examining properties of it is helpful to investigate properties of the Borda rule.

**Symmetry** For all $x, y \in \mathbb{R}^m_{++}$, $C(x,y) = C(y,x)$.

**Normalization** For all $x, y \in \mathbb{R}^m_{++}$, $0 \leq C(x,y) \leq 1$ and $C(x,x) = 1$.

**Scale invariance** For all $x, y \in \mathbb{R}^m_{++}$ and each $k > 0$, $C(kx, y) = C(x,y)$.

**Neutrality** For all $x, y \in \mathbb{R}^m_{++}$ and any permutation $\pi$ on $\{1,2,\ldots,m\}$, $C(\pi x, \pi y) = C(x,y)$, where $\pi x = (x_{\pi(j)})_{j=1}^m$.

These properties are not sufficient to characterize the cosine similarity operator. Characterizations of cosine similarity measure remains future research.

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