Logarithmically homogeneous preferences

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Abstract: A real-valued function on $\mathbb{R}^n_{++}$ is called logarithmically homogeneous if the function is given by a logarithmic transformation of a homogeneous function on $\mathbb{R}^n_{++}$. Specifying a consumer's preference on the consumption set $\mathbb{R}^n_{++}$ by a level comparison relation and a difference comparison relation, this paper provides some axioms on the two relations under which the full class of utility functions representing the relations are logarithmically homogeneous. It is also shown that all the utility functions are strongly concave and all the indirect utility functions are logarithmically homogeneous. Moreover, the additively-separable logarithmic utility functions are derived strengthening one of the axioms.

Key words: intensity comparison, logarithmically homogeneous utility function, additively-separable logarithmic utility function, Stone's price index.

JEL classification: D11, D60

1. Introduction

As a utility function exhibiting the *Marshallian constancy*, i.e., the marginal utility of income depends only on the income levels, Samuelson (1942, Equation (41)) introduces the following form of utility function[1]: a utility function U defined on the consumption set $X=\mathbb{R}^n_{++}$ is *logarithmically homogeneous* if and only if there is a 1-homogeneous function[2] u on X and two parameters $a > 0$ and b such that $U(\mathbf{x}) = a\cdot\log u(\mathbf{x}) + b$ for all $\mathbf{x} \in X$. In fact, since the (typical) indirect utility function corresponding to the logarithmically homogeneous utility function can be written as $V(\mathbf{p}, I) = \alpha\cdot\log[I/v(\mathbf{p})]+\beta$, where v is a (–1)-homogeneous function and $\alpha > 0$ and β are parameters, it holds that $\partial V(\mathbf{p}, I)/\partial I = \alpha\cdot[v(\mathbf{p})/I]\cdot[1/v(\mathbf{p})] = \alpha/I$, which means the Marshallian constancy.

Since all the logarithmically homogeneous utility functions U are *strongly concave* on X under the strict quasi-concavity assumption, and since the demand functions derived from the logarithmically homogeneous utility functions are 1-homogeneous with respect to prices, if we assume the logarithmically homogeneous utility functions for the market models in an applied welfare analysis, we can derive some normative conclusions (such as equity-regarding policy subscriptions) based on the strong concavity, while the market models with the homogeneous demand functions are simple and computationally tractable. In particular, an *additively-separable logarithmic* utility function: $U^*(\mathbf{x}) = a_1\cdot\log x_1 + \cdots + a_n\cdot\log x_n + b$ is a logarithmically homogeneous utility function of which 1-homogeneous function coincides with a Cobb-Douglas function, and then the competitive equilibria of the market models with the additively-separable logarithmic utilities can be computed using the corresponding Cobb-Douglas demand functions, $d(\mathbf{p}, I) = (a_1 I/p_1, \cdots, a_n I/p_n)$.

A characterization theorem of the representability by the 1-homogeneous utility function is given by Katzner (1970, Theorem 2.3-2), which shows that a (neo-classical) preference ordering satisfies the homogeneity axiom: $\mathbf{x} \sim \mathbf{y} \Rightarrow \lambda\cdot\mathbf{x} \sim \lambda\cdot\mathbf{y}$ for all $\lambda > 0$ if and only if there exists at least

[1] See also Katzner (1967) and Rader (1976). Mantel (1976) and Mas-Colell (1985, page 197) assume the utility functions to prove the Sonnenschein-Mantel-Debrue Theorem in the competitive market model.

[2] For a given integer r, a real-valued function f on X is called r-homogeneous if and only if f is homogeneous of degree r, i.e., $f(\lambda\cdot\mathbf{x}) = \lambda^r\cdot f(\mathbf{x})$ for all $\mathbf{x} \in X$.

one 1-homogeneous utility function representing the preference ordering.[3] Since the logarithmic function is monotone, it follows from Katzner's characterization result that the homogeneity axiom is necessary and sufficient for a preference ordering to be represented by a logarithmically homogeneous utility function. However, there exist some utility functions representing the preference ordering which are neither logarithmically homogeneous nor concave, even under the homogeneity axiom. Since a normative conclusion should not be derived from such a specific property of utility functions depending on the selection of utility functions in the applied welfare analysis, the preference foundation of the utility functions for the welfare analysis is not well-established by the characterization above. To avoid this phenomenon intrinsic in the ordinal utility theory, this paper adapts the cardinal utility theory based on the difference (intensity) comparisons to derive the logarithmically homogeneous utility functions,[4] i.e., a preference on the consumption set $\mathbb{R}^n_{++}$ is specified by a level comparison relation and a difference comparison relation,[5] and some axioms are provided for the preference under which the full class of utility functions representing the preference (including the difference comparison relation) are logarithmically homogeneous.

More concretely, the consumer's *preference* on the consumption set $X = \mathbb{R}^n_{++}$ is specified by a level comparison relation $\precsim$ and a difference comparison relation $\precsim_*$. The expression $\mathbf{x} \precsim \mathbf{y}$ means that $\mathbf{x}$ is preferred to $\mathbf{y}$, and the expression $(\mathbf{x} \to \mathbf{y}) \precsim_* (\mathbf{z} \to \mathbf{w})$ means that the difference from $\mathbf{x}$ to $\mathbf{y}$ is preferred to the difference from $\mathbf{z}$ to $\mathbf{w}$. A real-valued function f on the consumption set X is defined to be a *utility function representing the preference* if and only if not only the function f is order-preserving with respect to the level comparison relation $\precsim$, i.e.,

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[3] See also Dow and Werlang (1992), Candeal and Induráin (1995) and Bosi, Candeal and Induráin (2000).

[4] For the cardinal utility theory based on the difference comparisons, see Alt (1936), Shapley (1975), Köbberling (2006) and Miyake (2014).

[5] Generally, the growth rate comparison is used for the evaluation of the transitions of GDP, even in the international settings. The growth rate comparison can be recognized as a typical difference comparison relation. For example, if the t-year's annual growth rate of GDP, [GDP(t) – GDP(t–1)]/GDP(t–1) is greater than t*-year's growth rate of GDP, [GDP(t*) – GDP(t*–1)]/GDP(t*–1), then the transion [GDP(t–1) → GDP(t)] is preferred to the transion [GDP(t*–1) → GDP(t*)].

$f(\mathbf{x}) \geq f(\mathbf{y}) \Leftrightarrow \mathbf{x} \succsim \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in X$, but also the differences of f are order-preserving with respect to the difference comparison relation $\succsim_*$, i.e., $f(\mathbf{y}) - f(\mathbf{x}) \geq f(\mathbf{w}) - f(\mathbf{z}) \Leftrightarrow (\mathbf{x}\rightarrow \mathbf{y}) \succsim_* (\mathbf{z}\rightarrow \mathbf{w})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X$. Assuming that the level comparison relation satisfies the neo-classical conditions such as the monotonicity and strict quasi-convexity, it is shown that the full class of utility functions representing a preference are logarithmically homogeneous if and only if the preference satisfies the two axioms: one is the consistency axiom and the other one is the homogeneity axiom for the difference comparisons,[6] and that the utility functions are determined unique up to the positive affine transformations. Moreover, it is shown that the two axioms are necessary and sufficient for all the indirect utility functions to be logarithmically homogeneous, and that the additively-separable logarithmic utility functions are derived by strengthening the homogeneity axiom only in the axiomatic characterization above.[7]

The characterization results clarify the scope of applicability of logarithmically homogeneous utility functions. Specifically, since all the logarithmically homogeneous utility functions above are strongly concave, we can assume the logarithmic homogeneity and strongly concavity for the consumers' utility functions simultaneously, independent of the selections of utility functions.

The next section introduces the basic definitions and axiomatically derives the logarithmically homogeneous utility function, and the section 3 derives the logarithmically homogeneous indirect utility function. The section 4 derives the additively-separable logarithmic utility functions, and the section 5 proves the main results.

2. Logarithmically homogeneous utility functions

We introduce some concepts to specify a consumer's preferences on the consumption set, and provide a numerical representation theorem of the preference by means of logarithmically homogeneous utility functions.

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[6] The consistency axiom is a standard axiom introduced to ensure the existence of a utility function representing the preference $(\succsim, \succsim_*)$. This homogeneity axiom is a new axiom and strictly stronger than the homogeneity axiom for level comparison relation introduced by Katzner (1970).

[7] The strong homogeneity axiom is strictly stronger than the strong homogeneity axiom (budget-invariance axiom) for level comparison relations introduced by Trockel (1989), which characterizes the Cobb-Douglas preference orderings.

The consumption set X is defined by $X = \mathbb{R}^n_{++}$. A *level comparison relation* $\precsim$ is a complete and transitive binary relation on X satisfying the smoothness, monotonicity and strict quasi-convexity.[8] The expression $\mathbf{x} \succsim \mathbf{y}$ means that $\mathbf{x}$ is preferred to $\mathbf{y}$. The symmetric and asymmetric parts of $\succsim$ are denoted by $\sim$ and $\succ$, respectively. A *difference comparison relation* $\succsim_*$ on X is a complete and transitive quarternary relation on X.[9] The expression $(\mathbf{x} \to \mathbf{y}) \succsim_* (\mathbf{z} \to \mathbf{w})$ means that the transition (path) from $\mathbf{x}$ to $\mathbf{y}$ is preferred to the transition from $\mathbf{z}$ to $\mathbf{w}$. The symmetric and asymmetric parts of $\succsim_*$ are denoted by $\sim_*$ and $\succ_*$, respectively. A pair of a level comparison relation $\succsim$ and a difference comparison relation $\succsim_*$ on X is called a *preference* on X.

A real-valued function U on X is called a utility function representing $(\succsim, \succsim_*)$ if and only if the following two assertions hold:

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow U(\mathbf{x}) \geq U(\mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in X, \tag{1}$$

$$(\mathbf{x} \to \mathbf{y}) \succsim_* (\mathbf{z} \to \mathbf{w}) \Leftrightarrow U(\mathbf{y}) - U(\mathbf{x}) \geq U(\mathbf{w}) - U(\mathbf{z}) \text{ for all } \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X.^{10} \tag{2}$$

A utility function $U : X \to \mathbb{R}$ representing $(\succsim, \succsim_*)$ is *logarithmically homogeneous* if and only if U is C^2 on X and U satisfies the following condition:

$$U(\lambda \cdot \mathbf{x}) = \theta \cdot \log\lambda + U(\mathbf{x}) \text{ for all } \lambda > 0 \text{ and all } \mathbf{x} \in X, \tag{3}$$

where $\theta = U(e, \cdots, e) - U(1, \cdots, 1)$.[11] Another form of the condition (3) can be stated as follows: a utility function $U : X \to \mathbb{R}$ representing $(\succsim, \succsim_*)$ is *logarithmically homogeneous* if and only if

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[8] The definitions of these properties are given as follows: $\succsim$ is smooth iff $\succsim$ is represented by a C^2 utility function u and the determinant of the bordered Hessian of $u(\mathbf{x})$ is non-zero for all $\mathbf{x} \in X$; $\succsim$ is monotone iff $\mathbf{x} \gg \mathbf{y} \Rightarrow \mathbf{x} \succ \mathbf{y}$; $\succsim$ is strictly quasi-convex iff $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y} \Rightarrow \lambda \cdot \mathbf{x} + (1-\lambda) \cdot \mathbf{y} \succ \mathbf{y}$ for all $\lambda \in (0, 1)$.

[9] A quarternary relation $\succsim_*$ on X is complete iff $(\mathbf{x} \to \mathbf{y}) \succsim_* (\mathbf{z} \to \mathbf{w})$ or $(\mathbf{z} \to \mathbf{w}) \succsim_* (\mathbf{x} \to \mathbf{y})$ hold, and $\succsim_*$ is transitive iff $(\mathbf{x}^1 \to \mathbf{y}^1) \succsim_* (\mathbf{x}^2 \to \mathbf{y}^2)$ and $(\mathbf{x}^2 \to \mathbf{y}^2) \succsim_* (\mathbf{x}^3 \to \mathbf{y}^3) \Rightarrow (\mathbf{x}^1 \to \mathbf{y}^1) \succsim_* (\mathbf{x}^3 \to \mathbf{y}^3)$.

[10] This definition is introduced by Alt (1936).

[11] Setting $\mathbf{x} = (1, \cdots, 1)$ and $\lambda = e$ in (3), we have $U(e, \cdots, e) = \theta + U(1, \cdots, 1)$, which implies $\theta = U(e, \cdots, e) - U(1, \cdots, 1)$. This form of definition is standard in the mathematical programming theory. For example, see Nesterov and Nemirovskii (1994).

there is a C^2 and 1-homogeneous function $u : X \to \mathbb{R}_{++}$ and a real number b such that

$$U(\mathbf{x}) = \theta \cdot \log u(\mathbf{x}) + b \quad \text{for all } \mathbf{x} \in X, \tag{4}$$

where $\theta = U(e, \cdots, e) - U(1, \cdots, 1)$.[12] The two definitions (3) and (4) are equivalent. In fact, we have the following lemma:

Lemma 1: A utility function U representing $(\succsim, \succsim_*)$ satisfies (3) if and only if U satisfies (4).

We introduce the two axioms:

Consistency Axiom: If $(\mathbf{z} \to \mathbf{x}) \succsim_* (\mathbf{z} \to \mathbf{y})$ or $(\mathbf{y} \to \mathbf{z}) \succsim_* (\mathbf{x} \to \mathbf{z})$ for some $\mathbf{z} \in X$, then $\mathbf{x} \succsim \mathbf{y}$. Conversely, if $\mathbf{x} \succsim \mathbf{y}$, then $(\mathbf{z} \to \mathbf{x}) \succsim_* (\mathbf{z} \to \mathbf{y})$ and $(\mathbf{y} \to \mathbf{z}) \succsim_* (\mathbf{x} \to \mathbf{z})$ for all $\mathbf{z} \in X$.

Homogeneity Axiom: $(\mathbf{x} \to \mathbf{y}) \sim_* (\lambda \cdot \mathbf{x} \to \lambda \cdot \mathbf{y})$ for all $\lambda > 0$ and all $\mathbf{x}, \mathbf{y} \in X$.

The consistency axiom is introduced by Alt (1936) and Shapley (1975) to ensure the existence of utility function representing a preference $(\succsim, \succsim_*)$.[13] The homogeneity axiom above requires the difference comparison relation $\succsim_*$ to be homogeneous, i.e., if two differences are indifferent, then the indifference is invariant against changes of the unit of consumption goods which are common for all the goods, and this axiom is stronger than Katzner's (1970, Theorem 2.3-2) homogeneity axiom which requires the level comparison relation $\succsim$ to be homogeneous, i.e., the indifference relation $\sim$ is scale-invariant. Katzner's homogeneity axiom is introduced to ensure the existence of a 1-homogeneous utility function representing the level comparison relation. Formally, we have the following lemma:

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[12] Setting $\mathbf{x} = (e, \cdots, e)$ in (4), we have $U(e, \cdots, e) = \theta \cdot \log u(e, \cdots, e) + b$, and, setting $\mathbf{x} = (1, \cdots, 1)$ in (4), we have $U(1, \cdots, 1) = \theta \cdot \log u(1, \cdots, 1) + b$. Hence we have that $U(e, \cdots, e) - U(1, \cdots, 1) = \theta \cdot \log u(e, \cdots, e)/u(1, \cdots, 1) = \theta \cdot \log e = \theta$. This form of definition is introduced by Samuelson (1942) and Katzner (1967).

[13] Shapley (1975) shows that a preference $(\succsim, \succsim_*)$ is represented by a continuous utility function if and only if $(\succsim, \succsim_*)$ satisfies the consistency, continuity and crossover axioms. The continuity axiom requires $\{ (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) \in X \times X \times X \times X : (\mathbf{x} \to \mathbf{y}) \succsim_* (\mathbf{z} \to \mathbf{w}) \}$ to be relatively closed in $X \times X \times X \times X$ and the crossover axiom requires $(\mathbf{x} \to \mathbf{y}) \sim_* (\mathbf{z} \to \mathbf{w}) \Leftrightarrow (\mathbf{w} \to \mathbf{y}) \sim_* (\mathbf{z} \to \mathbf{x})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X$. It holds by Theorem 1 below the consistency and homogeneity axioms together imply the continuity and crossover axioms.

Lemma 2: If a preference $(\succsim, \succsim_*)$ satisfies the consistency and homogeneity axioms, then the following assertions hold:

(**i**) $\mathbf{x} \sim \mathbf{y} \Rightarrow \lambda\cdot\mathbf{x} \sim \lambda\cdot\mathbf{y}$ for all $\lambda > 0$ and all $\mathbf{x}, \mathbf{y} \in X$,[14]

(**ii**) There is a C^2 and 1-homogeneous function $u^* : X \to \mathbb{R}_{++}$ such that $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow u^*(\mathbf{x}) \geq u^*(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$ and that u^* is weakly concave on X.[15]

(**iii**) A function $v : X \to \mathbb{R}$ satisfies the 1-homogeneity and the condition: $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow v(\mathbf{x}) \geq v(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$ if and only if there exists $a > 0$ such that $v(\mathbf{x}) = a\cdot u^*(\mathbf{x})$ for all $\mathbf{x} \in X$.

The proof of Lemma 2 is given in Appendix A. Specifically, the existence of 1-homogeneous utility function u^* in Lemma 2 is shown by Katzner (1970, Theorem 2.3-2)[16] and the concavity of u^* is shown by Friedman (1973). Lemma 2(ii) and (iii) together imply that all the 1-homogeneous utility functions representing $\succsim$ are C^2, positive-valued and weakly concave on X.

It is obvious that the consistency and homogeneity axioms hold if there is a logarithmically homogeneous utility function. The following theorem implies that the two axioms are sufficient for the existence of the logarithmically homogeneous utility function:

Theorem 1: (**A**) The following statements for a preference $(\succsim, \succsim_*)$ are mutually equivalent:

(**i**) A preference $(\succsim, \succsim_*)$ satisfies the consistency and homogeneity axioms.

(**ii**) A preference $(\succsim, \succsim_*)$ is represented by a logarithmically homogeneous utility function.

(**iii**) There is a utility function representing $(\succsim, \succsim_*)$, and all the utility functions representing $(\succsim, \succsim_*)$ are logarithmically homogeneous and *strongly* concave on X.

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14 The assertion (i) of Lemma 2 is Katzner's homogeneity axiom. The difference comparison relations derived by the Cobb-Douglas utility functions do not satisfy our homogeneity axiom, but the Cobb-Douglas level comparison relations satisfy Katzner's homogeneity axiom. In fact, if $U(x_1,x_2) = (x_1x_2)^{1/2}$, it holds that $U(y_1,y_2) - U(x_1, x_2) = (y_1y_2)^{1/2} - (x_1x_2)^{1/2} < [(\lambda y_1\lambda y_2)^{1/2} - (\lambda x_1\lambda x_2)^{1/2}] = U(\lambda y_1, \lambda y_2) - U(\lambda x_1, \lambda x_2)$ for all $(y_1, y_2) \gg (x_1, x_2)$ and all $\lambda > 1$, and that $(y_1y_2)^{1/2} = (x_1x_2)^{1/2} \Rightarrow (\lambda y_1\lambda y_2)^{1/2} = (\lambda x_1\lambda x_2)^{1/2}$ for all $\lambda > 0$, which means that our homogeneity axiom is strictly stronger than Katzner's homogeneity axiom.

15 The *money-metric* utility function derived from $\succsim$ can be used for the 1-homogeneous utility function of $(\succsim, \succsim_*)$ satisfying the conditions in Lemma 2(ii). For the money-metric utility function, see Weymark (1985).

16 See also Dow and Werlang (1992, Proposition 1.5 and Theorem 1.7), Candeal and Induráin (1995, Section 4) and Bosi, Candeal and Induráin (2000).

(**B**) Suppose that a preference $(\precsim, \precsim_*)$ satisfies the consistency and homogeneity axioms, and let u be a 1-homogeneous function satisfying all the conditions in Lemma 2(ii). A real-valued function U on X is a utility function representing $(\precsim, \precsim_*)$ if and only if there exists $a > 0$ and b such that $U(\mathbf{x}) = a \cdot \log u(\mathbf{x}) + b$ for all $\mathbf{x} \in X$.

Theorem 1(A) implies that the two axioms are necessary and sufficient for the existence of a logarithmically homogeneous utility function and the two axioms are necessary and sufficient for all the utility functions to be logarithmically homogeneous and strongly concave. Theorem 1(A, B) implies that $\log u(\mathbf{x})$ in Theorem 1(B) is a logarithmically homogeneous utility function representing $(\precsim, \precsim_*)$ and that the logarithmically homogeneous utility functions are cardinal utility functions, i.e., the utility functions are determined unique up to the positive affine transformations. Specifically, it holds by Lemma 2(iii) that the formula: $U(\mathbf{x}) = a \cdot \log u(\mathbf{x}) + b$ is well-defined, independent of the selection of u.

The two axioms in Theorem 1(A) are mutually independent, which can be shown by constructing the counter examples. Define a preference $(\precsim^1, \precsim_*^1)$ by

$$\mathbf{x} \precsim^1 \mathbf{y} \Leftrightarrow \sum \log x_i \geq \sum \log y_i \text{ for all } \mathbf{x}, \mathbf{y} \in X, \text{ and}$$

$$(\mathbf{x} \to \mathbf{y}) \precsim_*^2 (\mathbf{z} \to \mathbf{w}) \Leftrightarrow \sum \log y_i - \sum \log x_i \leq \sum \log w_i - \sum \log z_i \text{ for all } \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X.$$

The preference $(\precsim^1, \precsim_*^1)$ satisfies the homogeneity axiom, but it does not satisfy the consistency axiom, which implies that the consistency axiom is independent. Define a difference comparison relation $\precsim_*^2$ by

$$(\mathbf{x} \to \mathbf{y}) \precsim_*^2 (\mathbf{z} \to \mathbf{w}) \Leftrightarrow \prod y_i - \prod x_i \geq \prod w_i - \prod z_i \text{ for all } \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X.$$

The preference $(\precsim^1, \precsim_*^2)$ satisfies the consistency axiom, since $(\precsim^1, \precsim_*^2)$ can be represented by $U(\mathbf{x}) = \prod x_i$, but it does not satisfy the homogeneity axiom, which implies that the homogeneity axiom is independent.

3. Logarithmically homogeneous indirect utility functions

In the previous section, a utility function U is defined to be representing a preference $(\precsim, \precsim_*)$ if and only if not only the function U is order-preserving with respect to the level comparison relation $\precsim$, but also the differences of U are order-preserving with respect to the difference comparison relation $\precsim_*$. Accordingly, this section defines the indirect utility function V of a preference $(\precsim, \precsim_*)$ by means of not only the level comparisons, but also the difference comparisons, and shows that the two axioms introduced in the previous section are necessary and sufficient for all the indirect utility functions to be logarithmically homogeneous.[17]

Let $(\precsim, \precsim_*)$ be a preference on $X = \mathbb{R}^n_{++}$, and let $D(\mathbf{p}, I)$ be the *demand set* of $(\mathbf{p}, I) \in \mathbb{R}^{n+1}_{++}$ defined by $D(\mathbf{p}, I) = \{ \mathbf{x} \in X\colon \mathbf{p}\,\mathbf{x} \le I \text{ and } \mathbf{x} \succsim \mathbf{y} \text{ for all } \mathbf{y} \in X \text{ with } \mathbf{p}\,\mathbf{y} \le I\}$.[18] Setting $B \equiv \{ (\mathbf{p}, I) \in \mathbb{R}^{n+1}_{++} : D(\mathbf{p}, I) \neq \emptyset \}$, we have the following lemma:

Lemma 3: (**i**) $B \neq \emptyset$; (**ii**) $D(\mathbf{p}, I)$ is a *singleton* for all $(\mathbf{p}, I) \in B$.

The *demand function* $d : B \to X$ can be defined by the unique element of $D(\mathbf{p}, I)$ for each $(\mathbf{p}, I) \in B$. The demand function $d(\mathbf{p}, I)$ has the following property:

Lemma 4: $d(\mathbf{p}, I)$ is C^1 on B.

A real-valued function V on B is called an *indirect* utility function of $(\precsim, \precsim_*)$ if and only if the following two conditions hold:

(**i**) $d(\mathbf{p}, I) \succsim d(\mathbf{q}, J) \Leftrightarrow V(\mathbf{p}, I) \ge V(\mathbf{q}, J)$ for all $(\mathbf{p}, I), (\mathbf{q}, J) \in B$,

(**ii**) $[\, d(\mathbf{p}, I) \to d(\mathbf{q}, J)\,] \succsim_* [\, d(\mathbf{p}^*, I^*) \to d(\mathbf{q}^*, J^*)]$

$$\Leftrightarrow V(\mathbf{q}, J) - V(\mathbf{p}, I) \ge V(\mathbf{q}^*, J^*) - V(\mathbf{p}^*, I^*) \quad \text{for all } (\mathbf{p}, I), (\mathbf{q}, J), (\mathbf{p}^*, I^*), (\mathbf{q}^*, J^*) \in B.$$

Specifically, an indirect utility function $V^* : B \to \mathbb{R}$ of $(\precsim, \precsim_*)$ is called *logarithmically homogeneous* (*with respect to price vectors*) if and only if the following conditions hold:

(**iii**) $(\lambda\mathbf{p}, I) \in B$ and $(\mathbf{p}, \lambda I) \in B$ for all $(\mathbf{p}, I) \in B$ and all $\lambda > 0$.

(**iv**) There is a C^1 and (–1)-homogeneous function $v : P^* \to \mathbb{R}_{++}$ and two real numbers $\alpha > 0$ and β such that $V^*(\mathbf{p}, I) = \alpha \cdot \log v(p_1/I, \cdots, p_n/I) + \beta$ for all $(\mathbf{p}, I) \in B$, where $P^* \equiv \{ p \in P\colon (\mathbf{p}, I) \in B$ for some $I > 0 \}$.

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[17] Mas-Colell (1985, Proof of Proposition 5.5.8, page 197) introduces a logarithmically homogeneous indirect utility function to prove the Sonnenschein-Mantel-Debrue Theorem.

[18] For a case of $D(p, I) = \emptyset$, see Mas-Colell (1985, Figure 2.7.1, page 85).

In fact, if an indirect utility function V^* of $(\precsim, \precsim^*)$ is logarithmically homogeneous, it holds that $V^*(\lambda\cdot\mathbf{p}, I) = -\alpha\cdot\log\lambda + V^*(\mathbf{p}, I)$ for all $(\mathbf{p}, I) \in B$ and all $\lambda > 0$, which means that the definition of the logarithmic homogeneity for the indirect utility functions is consistent to the definition for direct utility functions given by (3). The main results of this section is the following theorem:

Theorem 2: (**A**) The following statements for a preference $(\precsim \precsim_*)$ are mutually equivalent:

(**i**) A preference $(\precsim, \precsim_*)$ satisfies the consistency and homogeneity axioms.

(**ii**) There is a logarithmically homogeneous indirect utility function of $(\precsim, \precsim_*)$.

(**iii**) There is an indirect utility function of $(\precsim, \precsim_*)$, and all the indirect utility functions of $(\precsim, \precsim_*)$ are logarithmically homogeneous on B.

(**B**) Suppose that a preference $(\precsim, \precsim_*)$ on X satisfies the consistency and homogeneity axioms, and let u be a 1-homogeneous function on X satisfying all the conditions in Lemma 2(ii). A real-valued function V on B is an indirect utility function of $(\precsim, \precsim_*)$ if and only if there exists $\alpha > 0$ and β such that $V(\mathbf{p}, I) = \alpha\cdot\log u(d(\mathbf{p}, I)) + \beta$ for all $(\mathbf{p}, I) \in B$.[19]

Theorem 2(A) implies that the two axioms are necessary and sufficient for the existence of a logarithmically homogeneous indirect utility function and the two axioms are necessary and sufficient for all the indirect utility functions to be logarithmically homogeneous, or to satisfy the Marshallian constancy. Thus, we can assume a logarithmically homogeneous indirect utility function for the consumer whose preference satisfies the two axioms above, since all indirect utility functions of the consumer are logarithmically homogeneous. Theorem 2(A, B) implies that $\log u(d(\mathbf{p},I))$ is an indirect utility function of $(\precsim,\precsim_*)$ and that the logarithmically homogeneous indirect utility functions are cardinal utility functions, i.e., the indirect utility functions are determined unique up to the positive affine transformations. Specifically, the *money-metric* indirect utility function derived from $\precsim$ can be used for the (–1)-homogeneous indirect utility

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[19] Setting $v(\mathbf{t}) = u(d(\mathbf{t}, 1))$ for all $\mathbf{t} \in P^*$, it holds that $v(\lambda\mathbf{t}) = u(d(\lambda\mathbf{t}, 1)) = u(\lambda^{-1}\cdot d(\mathbf{t}, 1)) = \lambda^{-1}\cdot u(d(\mathbf{t}, 1)) = \lambda^{-1}\cdot v(\mathbf{t})$ and that $\log v(p_1/I, \cdots, p_n/I) = \log u(d(p_1/I, \cdots, p_n/I, 1)) = \log u(Id(\mathbf{p}, 1)) = \log u(d(\mathbf{p}, I)) = V(\mathbf{p}, I)$, which implies that $v(\mathbf{t}) = u(d(\mathbf{t}, 1))$ is the (–1)-homogeneous function in the definition of the logarithmically homogeneous indirect utility function.

function of $(\succsim, \succsim_*)$.[20]

Since u in Theorem 2(B) is a 1-homogeneous function on X satisfying the conditions in Lemma 2, the typical indirect utility function $V(\mathbf{p}, I)$ of $(\succsim, \succsim_*)$ can be written as

$$V(\mathbf{p}, I) = \log I + \log u(d(\mathbf{p}, 1)) \quad \text{for all } (\mathbf{p}, I) \in B, \tag{5}$$

which means that the consumer's cardinal utility values of $V(\mathbf{p}, I)$ can be decomposed into the income part, $\log I$ and price part, $\log u(d(\mathbf{p}, 1))$, as long as the consumer's preference $(\succsim, \succsim_*)$ satisfies the two axioms. Alternatively, setting $c(\mathbf{p}) = 1/u(d(\mathbf{p}, 1))$ in (5), the typical indirect utility function $V(\mathbf{p}, I)$ of $(\succsim, \succsim_*)$ can be written as

$$V(\mathbf{p}, I) = [\, \log I/c(\mathbf{p}) \,] \quad \text{for all } (\mathbf{p}, I) \in B.$$

For a consumer satisfying the two axioms above, $I/c(\mathbf{p})$ and $c(\mathbf{p})$ can be recognized as a normalized income level and a (personalized) deflator, respectively, since $V(\mathbf{p}, I)$ is a cardinal indirect utility function of the consumer. When the price vector is fixed, the indirect utility function V can be recognized as the logarithmic utility function on income levels as considered by Mandelbrot (1960, Section 1.5), Graff (2014) and Miyake (2014).

4. Additively-separable logarithmic utility functions

As the utility functions not only exhibiting the Marshallian constancy, but also satisfying the additive separability, Samuelson (1942, Equation (34)) introduces the following form of real-valued function on the consumption set X: a real-valued function f on X is *additively-separable logarithmic* if and only if there are n+1 real numbers $a_1 > 0, a_2 > 0, \cdots, a_n > 0$ and b such that:

$$f(\mathbf{x}) = a_1 \log x_1 + a_2 \log x_2 + \cdots + a_n \log x_n + b \quad \text{for all } \mathbf{x} \in X.$$

If a utility function U representing a preference $(\succsim, \succsim_*)$ on X is additively-separable logarithmic, then U is a logarithmically homogeneous utility function.

As the form of the indirect utility functions corresponding to the additively-separable logarithmic utility functions on X, we introduce the following form of real-valued function on the space of budgets $\mathbb{P} \times \mathbb{I} \equiv \mathbb{R}^{n+1}_{++}$: a real-valued function g on $\mathbb{P} \times \mathbb{I}$ is *additively-separable logarithmic* if and only if there are n+2 real numbers $\alpha_0 > 0, \alpha_1 < 0, \alpha_2 < 0, \cdots, \alpha_n < 0$ $(\sum \alpha_i = -\alpha_0)$ and

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[20] For the money-metric indirect utility function, see Weymark (1985, Section 5).

β such that

$$g(\mathbf{p}, I) = \alpha_0 \log I + \alpha_1 \log p_1 + \alpha_2 \log p_2 + \cdots + \alpha_n \log p_n + \beta \quad \text{for all } (\mathbf{p}, I) \in \mathbb{P}\times\mathbb{I}.$$

If an indirect utility function V of $(\succsim, \succsim_*)$ on $\mathbb{P}\times\mathbb{I}$ is additively-separable logarithmic, then U is a logarithmically homogeneous indirect utility function. In fact, setting

$$v(t_1,\cdots, t_n) = \alpha_1 \log t_1 + \alpha_2 \log t_2 + \cdots + \alpha_n \log t_n \quad \text{for all } (t_1,\cdots, t_n) \in \mathbb{P},$$

it holds that v is C^1 and (-1)-homogeneous and satisfies $g(\mathbf{p}, I) = \log v(p_1/I, \cdots, p_n/I) + \beta$ for all $(\mathbf{p}, I) \in B$. This section provides an axiomatic characterization theorem for both forms of utility functions. For the theorem, we need a definition:

Strong homogeneity axiom: $(\mathbf{x} \to \mathbf{y}) \sim_* (\mathbf{c}*\mathbf{x} \to \mathbf{c}*\mathbf{y})$ for all $\mathbf{c} \in X$, where $\mathbf{c}*\mathbf{x} = (c_1\cdot x_1, c_2\cdot x_2, \cdots, c_n\cdot x_n) \in X$.

If $c_1 = c_2 = \cdots = c_n$ in the statement of the strong homogeneity axiom, the axiom coincides with the homogeneity axiom, and then the strong homogeneity axiom is stronger than the homogeneity axiom.[21] Moreover, the strong homogeneity axiom implies the strong homogeneity of the level comparison relation which ensures the existence of a Cobb-Douglas utility function representing $\succsim$. Formally, we have the following lemma:

Lemma 5: If a preference $(\succsim, \succsim_*)$ satisfies the consistency and strong homogeneity axioms, then the following assertions hold:

(**i**) $\mathbf{x} \sim \mathbf{y} \Rightarrow \mathbf{c}*\mathbf{x} \sim \mathbf{c}*\mathbf{y}$ for all $\mathbf{c} \in X$.

(**ii**) $\mathbf{x} \succsim \mathbf{y} \Rightarrow \mathbf{c}*\mathbf{x} \succsim \mathbf{c}*\mathbf{y}$ for all $\mathbf{c} \in X$.

(**iii**) There is a function $w : X \to \mathbb{R}$ such that $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow w(\mathbf{x}) \geq w(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$ and that $w(\mathbf{x}) = x_1^{a_1} \cdots x_n^{a_n}$ for some n positive numbers: $a_1 > 0,\cdots, a_n > 0$ satisfying $a_1 + \cdots + a_n = 1$. Moreover, the demand function of $(\succsim, \succsim_*)$ is given by $d(\mathbf{p}, I) = (a_1 I/p_1, \cdots, a_n I/p_n)$.

The proof of Lemma 5(i, ii) is given in Appendix A. Specifically, in order to ensures the existence of the Cobb-Douglas utility function Trockel (1989) introduces the condition (ii) in Lemma 5, which

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[21] Since the difference comparison relations derived by the Cobb-Douglas utility functions do not satisfy the homogeneity axiom as shown in Footnote 14 in Section 2 of this paper, the difference comparison relations derived by the Cobb-Douglas utility functions do not satisfy the strong homogeneity axiom.

is called the budget invariance axiom. The conditions (i) and (ii) in Lemma 5 require the level comparison relations to be coordinate-wisely homogeneous and the two conditions mean the strong homogeneity of the level comparison relations, while our strong homogeneity axiom above requires the difference comparison relation to be coordinate-wisely homogeneous, i.e., the indifference relation $\sim_*$ on the differences is invariant against changes of the units of consumption goods specific to each of the goods. The main result of this section is the following theorem:

Theorem 3: (**A**) The following statements for a preference $(\precsim, \precsim_*)$ on X are mutually equivalent:

(**i**) A preference $(\precsim, \precsim_*)$ on X satisfies the consistency and strong homogeneity axioms.

(**ii**) A preference $(\precsim, \precsim_*)$ on X is represented by an additively-separable logarithmic function on X.

(**iii**) There is a utility function on X representing $(\precsim, \precsim_*)$, and all the utility functions representing $(\precsim, \precsim_*)$ are additively-separable logarithmic functions on X.

(**iv**) There is an additively-separable logarithmic function on $\mathbb{P}\times\mathbb{I}$ which is an indirect utility function of $(\precsim, \precsim_*)$.

(**v**) There is an indirect utility function on $\mathbb{P}\times\mathbb{I}$ of $(\precsim, \precsim_*)$, and all the indirect utility functions of $(\precsim, \precsim_*)$ are additively-separable logarithmic on $\mathbb{P}\times\mathbb{I}$.

(**B**) Suppose that a preference $(\precsim, \precsim_*)$ on X satisfies the consistency and strong homogeneity axioms, and let w be the Cobb-Douglas utility function of $(\precsim, \precsim_*)$ in Lemma 5, i.e., $w(\mathbf{x}) = x_1^{a_1} \cdots x_n^{a_n}$ for some n positive numbers: $a_1 > 0, \cdots, a_n > 0$ satisfying $a_1 + \cdots + a_n = 1$. Then the following assertions hold:

(**vi**) A real-valued function U on X is a utility function representing $(\precsim, \precsim_*)$ if and only if there exists $\alpha > 0$ and β such that

$$U(\mathbf{x}) = \alpha\cdot(a_1\cdot\log x_1 + a_2\cdot\log x_2 + \cdots + a_n\cdot\log x_n) + \beta \quad \text{for all } \mathbf{x} \in X.$$

(**vii**) A real-valued function V on $\mathbb{P}\times\mathbb{I}$ is an indirect utility function of $(\precsim, \precsim_*)$ if and only if there exists $\alpha > 0$ and β such that

$$\begin{aligned} V(\mathbf{p}, I) &= \alpha\cdot[(-a_1)\cdot\log (p_1/I) + (-a_2)\cdot\log (p_2/I) + \cdots + (-a_n)\cdot\log (p_n/I)] + \beta \\ &= \alpha\cdot[\log I - (a_1\cdot\log p_1 + a_2\cdot\log p_2 + \cdots + a_n\cdot\log p_n)] + \beta \quad \text{for all } (\mathbf{p}, I) \in \mathbb{P}\times\mathbb{I}. \end{aligned}$$

It holds by Theorem 3(B) that, if a preference $(\precsim, \precsim_*)$ satisfies the consistency and strong homogeneity axioms, not only a utility function representing $(\succsim, \succsim_*)$, but also an indirect utility function of $(\succsim, \succsim_*)$ are determined unique up to the positive affine transformations. Since $(a_1 \log p_1 + \cdots + a_n \log p_n)$ in Theorem 3(vii) is called *Stone's price index*, Theorem 3 implies that the welfare loss (costs) due to the price changes should be measured by the index for the consumers whose preferences satisfy the two axioms. Moreover, $V(\mathbf{p}, I)$ can be written as

$$V(\mathbf{p}, I) = \log (I/p_1^{a_1} \cdots p_n^{a_n}) \quad \text{for all } (\mathbf{p}, I) \in \mathbb{P}\times\mathbb{I}.$$

Hence, if a (representative) consumer satisfies the two axioms above, $p_1^{a_1} \cdots p_n^{a_n}$ can be recognized as a deflator, since $V(\mathbf{p}, I)$ is a cardinal indirect utility function of the consumer.

5. Proof of the theorems

Proof of Theorem 1: (**A**) It can be shown easily that (iii) ⇒ (ii) ⇒ (i). We will prove that (i) ⇒ (ii), and then prove that (ii) ⇒ (iii).

[(i) ⇒ (ii)]: Suppose that a preference $(\succsim, \succsim_*)$ satisfies the consistency and homogeneity axioms. It holds by Lemma 2 that there exists a C^2 and 1-homogeneous utility function $v : X \to \mathbb{R}_{++}$ representing $\succsim$. Define $U(\mathbf{x}) = \log v(\mathbf{x})$ for all $\mathbf{x} \in X$. Since *log*(·) is monotone, we have that

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \log v(\mathbf{x}) \geq \log v(\mathbf{y}) \Leftrightarrow U(\mathbf{x}) \geq U(\mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in X. \tag{6}$$

Denote $\mathbf{e} = (1, 1, \cdots, 1)$. Fix any $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X$. Since $\succsim$ is neo-classical, there are $t, s, z, w \in \mathbb{R}_{++}$ such that

$$\mathbf{x} \sim t\cdot\mathbf{e}, \ \mathbf{y} \sim s\cdot\mathbf{e}, \mathbf{z} \sim z\cdot\mathbf{e} \text{ and } \mathbf{w} \sim w\cdot\mathbf{e}, \tag{7}$$

which implies that

$$v(\mathbf{x}) = v(t\cdot\mathbf{e}) = t\cdot v(\mathbf{e}), v(\mathbf{y}) = s\cdot v(\mathbf{e}), v(\mathbf{z}) = z\cdot v(\mathbf{e}) \text{ and } v(\mathbf{w}) = w\cdot v(\mathbf{e}). \tag{8}$$

Lemma 6: (**i**) If $\mathbf{x} \sim \mathbf{x}^*$, $\mathbf{y} \sim \mathbf{y}^*$, $\mathbf{z} \sim \mathbf{z}^*$, $\mathbf{w} \sim \mathbf{w}^*$, then $(\mathbf{x} \to \mathbf{y}) \succsim_* (\mathbf{z} \to \mathbf{w}) \Leftrightarrow (\mathbf{x}^* \to \mathbf{y}^*) \succsim_* (\mathbf{z}^* \to \mathbf{w}^*)$ for all $\mathbf{x}, \mathbf{y}, \mathbf{x}^*, \mathbf{y}^*, \mathbf{z}, \mathbf{w}, \mathbf{z}^*, \mathbf{w}^* \in X$.

(**ii**) $\log s - \log t \geq \log w - \log z \Leftrightarrow (t\cdot\mathbf{e} \to s\cdot\mathbf{e}) \succsim_* (z\cdot\mathbf{e} \to w\cdot\mathbf{e})$.

It holds by (7) and Lemma 6(i, ii) and (8) that

$$(\mathbf{x} \to \mathbf{y}) \succsim_* (\mathbf{z} \to \mathbf{w}) \Leftrightarrow (t\cdot\mathbf{e} \to s\cdot\mathbf{e}) \succsim_* (z\cdot\mathbf{e} \to w\cdot\mathbf{e}) \Leftrightarrow \log s - \log t \geq \log w - \log z$$

$$\Leftrightarrow (\log s - \log t)\cdot v(\mathbf{e}) \geq (\log w - \log z)\cdot v(\mathbf{e}) \Leftrightarrow \log s\cdot v(\mathbf{e}) - \log t\cdot v(\mathbf{e}) \geq \log w\cdot v(\mathbf{e}) - \log z\cdot v(\mathbf{e})$$

$$\Leftrightarrow \log v(\mathbf{y}) - \log v(\mathbf{x}) \geq \log v(\mathbf{w}) - \log v(\mathbf{z}) \;\Leftrightarrow\; U(\mathbf{y}) - U(\mathbf{x}) \geq U(\mathbf{w}) - U(\mathbf{z}). \quad (9)$$

Thus it holds by (6) and (9) that $U(\mathbf{x}) = \log v(\mathbf{x})$ is a logarithmically homogeneous utility function representing $(\succsim, \succsim_*)$.

[(ii) $\Rightarrow$ (iii)]: Suppose that there is a logarithmically homogeneous utility function U^* representing $(\succsim, \succsim_*)$, which implies that $(\succsim, \succsim_*)$ satisfies the consistency and homogeneity axioms. Let U be a utility function representing $(\succsim, \succsim_*)$. We need a lemma:

Lemma 7: There exists $a > 0$ and b such that $U(\mathbf{x}) = a\cdot U^*(\mathbf{x}) + b$ for all $\mathbf{x} \in X$.

Then it holds by Lemma 7 that U is logarithmically homogeneous on X. Since the logarithmic function is strongly concave, and since the 1-homogeneous function u in (4) is weakly concave on X by Lemma 5(ii), U is strongly concave on X.

(**B**) Suppose that a preference $(\succsim, \succsim_*)$ on X satisfies the consistency and homogeneity axioms, and let u be a 1-homogeneous function on X satisfying all the conditions in Lemma 2. It holds by the arguments above [(i) $\Rightarrow$ (ii)] that $U^*(\mathbf{x}) = \log u(\mathbf{x})$ is a logarithmically homogeneous utility function representing $(\succsim, \succsim_*)$. If U is a utility function representing $(\succsim, \succsim_*)$, then it holds by Lemma 7 that there exists $a > 0$ and b such that

$$U(\mathbf{x}) = a\cdot U^*(\mathbf{x}) + b = a\cdot \log u(\mathbf{x}) + b \quad \text{for all } \mathbf{x} \in X. \quad (10)$$

Conversely, if (10) holds for a function $U : X \rightarrow \mathbb{R}$, then U represents $(\succsim, \succsim_*)$. □

Proof of Theorem 2: (**A**) [(i) $\Rightarrow$ (ii)] Let u be a 1-homogeneous function satisfying all the conditions in Lemma 2.

Lemma 8: Define a function V^* on B by $V^*(\mathbf{p}, I) = \log u(d(\mathbf{p}, I))$ for all $(\mathbf{p}, I) \in B$. Then V^* is an indirect utility function of $(\succsim, \succsim_*)$.

Moreover, V^* is logarithmically homogeneous, since $v(\mathbf{p}) \equiv u(d(\mathbf{p}, 1))$ is a C^1 and (–1)-homogeneous function on P^*. In fact, it holds by Lemma 4 that $u(d(\mathbf{p}, 1))$ is C^1 on P^*. We need a lemma:

Lemma 9: (**A**) $B = P^* \times \mathbb{R}_{++}$; (**B**) For all $(\mathbf{p}, I) \in B$ and all $\lambda > 0$, the following assertions hold: (**i**) $(\lambda\mathbf{p}, I) \in B$ and $d(\lambda\mathbf{p}, I) = (1/\lambda)\cdot d(\mathbf{p}, I)$; (**ii**) $(\mathbf{p}, \lambda I) \in B$ and $d(\mathbf{p}, \lambda I) = \lambda\cdot d(\mathbf{p}, I)$.

It holds by Lemma 9(Bi) that $v(\lambda \cdot \mathbf{p}) = u(d(\lambda \cdot \mathbf{p}, 1)) = u((1/\lambda) \cdot d(\mathbf{p}, 1)) = (1/\lambda) \cdot v(\mathbf{p})$ for all $\lambda > 0$ and all $\mathbf{p} \in P^*$. It holds by Lemma 9(Bii) that $V^*(\mathbf{p}, I) = \log u(d(\mathbf{p}, I)) = \log u(I \cdot d(\mathbf{p}, 1)) = \log u(d((1/I)\mathbf{p}, 1)) = \log v(p_1/I, \cdots, p_n/I)$ for all $(\mathbf{p}, I) \in B$.

[(ii) ⇒ (i)] Suppose that $V^*(\mathbf{p}, I) = a \cdot \log v(p_1/I, \cdots, p_n/I) + b$ is a logarithmically homogeneous indirect utility function of $(\succsim, \succsim_*)$. Since v is (-1)-homogeneous, $\log I + \log v(\mathbf{p})$ is an indirect utility function of $\succsim$. It holds by this and Roy's identity that $d(\mathbf{p}, \lambda I) = \lambda d(\mathbf{p}, I)$ for all $(\mathbf{p}, I) \in B$ and all $\lambda > 0$, which implies that the original preference $(\succsim, \succsim_*)$ satisfies the homogeneity axiom. It is obvious that $(\succsim, \succsim_*)$ satisfies the consistency axiom.

[(iii) ⇒ (ii)] Obvious.

[(ii) ⇒ (iii)] Suppose that there is a logarithmically homogeneous indirect utility function $V^* : B \to \mathbb{R}$ of $(\succsim, \succsim_*)$. Since [(ii) ⇒ (i)] holds, $(\succsim, \succsim_*)$ satisfies the two axioms. Let $V : B \to \mathbb{R}$ be an indirect utility function of $(\succsim, \succsim_*)$.

Lemma 10: There exists $\alpha > 0$ and β such that $V(\mathbf{p}, I) = \alpha \cdot V^*(\mathbf{p}, I) + \beta$ for all $(\mathbf{p}, I) \in B$.

It holds by Lemma 10 that V is logarithmically homogeneous.

(**B**) Suppose that a preference $(\succsim, \succsim_*)$ on X satisfies the consistency and homogeneity axioms, and let u be a C^2 and 1-homogeneous function on X satisfying the conditions in Lemma 2. It holds by Lemma 8 that $V^*(\mathbf{p}, I) = \log u(d(\mathbf{p}, I))$ is an indirect utility function of $(\succsim, \succsim_*)$. If $V : B \to \mathbb{R}$ be an indirect utility function of $(\succsim, \succsim_*)$, then it holds by Lemma 10 that there exists $\alpha > 0$ and β such that

$$V(\mathbf{p}, I) = \alpha \cdot [\, \log u(d(\mathbf{p}, I)) \,] + \beta = \alpha \cdot [\, \log I + \log u(d(\mathbf{p}, 1)) \,] + \beta \quad \text{for all } (\mathbf{p}, I) \in B. \qquad (11)$$

Conversely, if (11) holds for a function $V : B \to \mathbb{R}$, then V is an indirect utility function of $(\succsim, \succsim_*)$.

□

Proof of Theorem 3: (**A**) Since $\succsim$ is smooth, there is a C^2 utility function u representing $\succsim$. Hence, for each $\mathbf{x} \in X$, setting $(\mathbf{p}, I) = (\nabla u(\mathbf{x}), \nabla u(\mathbf{x}) \cdot \mathbf{x}) \in \mathbb{P} \times \mathbb{I}$, it holds that $\mathbf{x} = d(\mathbf{p}, I)$. Using this fact, it can be shown easily that (iii) ⇒ (i) and (v) ⇒ (i). We will prove that (i) ⇒(ii) ⇒(iii), and then prove that (i) ⇒ (iv) ⇒ (v).

[(i) ⇒ (ii)]: Suppose that a preference $(\succsim, \succsim_*)$ satisfies the consistency and strong homogeneity axioms. It holds by Lemma 5(iii) that there exists a Cobb-Douglas utility function representing

$\precsim$, $w(\mathbf{x}) = x_1^{a_1} \cdots x_n^{a_n}$ $(a_1 > 0, \cdots, a_n > 0, a_1 + \cdots + a_n = 1)$. Define $v(\mathbf{x})$ by $v(\mathbf{x}) = \log w(\mathbf{x}) = a_1 \log x_1 + \cdots + a_n \log x_n$ for all $\mathbf{x} \in X$. Since $\log(\cdot)$ is monotone, it holds that $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow v(\mathbf{x}) \geq v(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$. Since $w(\mathbf{x})$ is 1-homogeneous, it holds by Theorem 1 that

$$(\succsim, \succsim_*) \text{ is represented by } v(\mathbf{x}) = \log w(\mathbf{x}) = a_1 \log x_1 + \cdots + a_n \log x_n . \tag{12}$$

[(ii) $\Rightarrow$ (iii)]: Suppose that there is a utility function U representing $(\succsim, \succsim_*)$. It holds by Theorem 1(B) and (12) that there exists $a > 0$ and b such that $U(\mathbf{x}) = av(\mathbf{x}) + b$ for all $\mathbf{x} \in X$, which implies that U is additively-separable logarithmic on X.

[(i) $\Rightarrow$ (iv)]: Suppose that $(\succsim, \succsim_*)$ satisfies the consistency and strong homogeneity axioms. Since $w(\mathbf{x})$ is 1-homogeneous, it holds by Theorem 2(B) that

$$\begin{aligned} V^*(\mathbf{p}, I) &= \log w(d(\mathbf{p}, I)) \\ &= \log w(a_1 I/p_1, \cdots, a_n I/p_n) = a_1 \log (a_1 I/p_1) + \cdots + a_n \log (a_n I/p_n) \end{aligned} \tag{13}$$

is an indirect utility function of $(\succsim, \succsim_*)$.

[(iv) $\Rightarrow$ (v)]: Let V be an indirect utility function of $(\succsim, \succsim_*)$. It holds by Theorem 2(B) that there exists $a > 0$ and b such that $V(\mathbf{p}, I) = aV^*(\mathbf{p}, I) + b$ for all $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$. Then V is additively-separable logarithmic.

(**B**) Suppose that $(\succsim, \succsim_*)$ satisfies the consistency and strong homogeneity axioms. The assertion (vi) is a direct consequence of (12) and Theorem 1(B), and the assertion (vii) is a direct consequence of (13) and Theorem 2(B). □

Appendix A

Appendix A proves all the lemmas of this paper. We need a claim proved in Appendix B:

Claim 1: If $(\succsim, \succsim_*)$ satisfies the consistency and homogeneity axioms, then the following assertions hold:

(**i**) $\mathbf{x} \sim \mathbf{y} \Rightarrow (\mathbf{z} \to \mathbf{x}) \sim_* (\mathbf{z} \to \mathbf{y})$ for all $\mathbf{z} \in X$.

(**ii**) $\mathbf{x} \succ \mathbf{y} \Rightarrow (\mathbf{z} \to \mathbf{x}) \succ_* (\mathbf{z} \to \mathbf{y})$ for all $\mathbf{z} \in X$.

(**iii**) $\log a - \log b \geq \log a^* - \log b^* \Leftrightarrow (b \cdot \mathbf{e} \to a \cdot \mathbf{e}) \succsim_* (b^* \cdot \mathbf{e} \to a^* \cdot \mathbf{e})$ for all $a, b, a^*, b^* \in \mathbb{R}_{++}$.

(**iv**) Suppose that f is a real-valued function on $\mathbb{R}_{++}$ satisfying the two conditions: $f(t) \geq f(s) \Leftrightarrow t \cdot \mathbf{e} \succsim s \cdot \mathbf{e}$ for all $t, s \in \mathbb{R}_{++}$; $f(s) - f(t) \geq f(w) - f(z) \Leftrightarrow (t \cdot \mathbf{e} \to s \cdot \mathbf{e}) \succsim_* (z \cdot \mathbf{e} \to w \cdot \mathbf{e})$ for all $t, s, z, w \in$

$\mathbb{R}_{++}$, and suppose that g is another real-valued function on $\mathbb{R}_{++}$ satisfying the two conditions above. Then there exists $a > 0$ and b such that $g(t) = a \cdot f(t) + b$ for all $t \in \mathbb{R}_{++}$.

Proof of Lemma 1: If a utility function U representing $(\succsim, \succsim_*)$ satisfies (4), then U satisfies (3), since u is 1-homogeneous. Conversely, if a utility function U representing $(\succsim, \succsim_*)$ satisfies (3), then U satisfies (4), by setting $u(\mathbf{x}) = [\, e^{U(\mathbf{x})} \,]^{1/\theta}$ and $b = 0$, where $\theta = U(e,\cdots,e) - U(1,\cdots,1)$. In fact, $\theta \cdot \log u(\mathbf{x}) + b = \theta \cdot \log [\, e^{U(\mathbf{x})} \,]^{1/\theta} + 0 = \theta \cdot (1/\theta) \cdot U(\mathbf{x}) \cdot \log e = U(\mathbf{x})$ for all $\mathbf{x} \in X$. □

Proof of Lemma 2: (**i**) Suppose that $\mathbf{x} \sim \mathbf{y}$ and $\lambda \cdot \mathbf{x} \succ \lambda \cdot \mathbf{y}$ for some $\lambda > 0$ and some $\mathbf{x}, \mathbf{y} \in X$. It holds by Claim 1 that

$$(\mathbf{x} \to \mathbf{x}) \sim_* (\mathbf{x} \to \mathbf{y}) \quad \text{and} \quad (\lambda \cdot \mathbf{x} \to \lambda \cdot \mathbf{x}) \succ_* (\lambda \cdot \mathbf{x} \to \lambda \cdot \mathbf{y}). \tag{A1}$$

It holds by homogeneity axiom that

$$(\mathbf{x} \to \mathbf{y}) \sim_* (\lambda \cdot \mathbf{x} \to \lambda \cdot \mathbf{y}) \tag{A2}$$

and

$$(\mathbf{x} \to \mathbf{x}) \sim_* (\lambda \cdot \mathbf{x} \to \lambda \cdot \mathbf{x}). \tag{A3}$$

Hence it holds by (A1) and (A2) that $(\lambda \cdot \mathbf{x} \to \lambda \cdot \mathbf{x}) \succ_* (\lambda \cdot \mathbf{x} \to \lambda \cdot \mathbf{y}) \sim_* (\mathbf{x} \to \mathbf{y}) \sim_* (\mathbf{x} \to \mathbf{x})$, which contradicts with (A3).

(**ii**) The existence of a 1-homogeneous utility function $u : X \to \mathbb{R}_{++}$ representing $\succsim$ can be proved by Lemma 2(i) and Dow and Werlang (1992, Proposition 1.5 and Theorem 1.7). The positiveness of u is proved by the 1-homogeneity and monotonicity of u. In fact, if $u(\mathbf{x}) \leq 0$ for some $\mathbf{x} \in X$, then $u(2\mathbf{x}) \leq u(\mathbf{x})$ holds by the 1-homogeneity, which contradicts with the monotonicity. Since u represents $\succsim$, and since $\succsim$ satisfies the neo-classical conditions, u is monotone and strictly quasi-concave. Since u is 1-homogeneous, it holds that

$$u(x, \cdots, x) = x \cdot u(1, \cdots, 1) \quad \text{for all } x > 0, \tag{A4}$$

which implies that $u(x, x)$ is C^2 with respect to x. Since $\succsim$ is smooth, there exists a C^2 utility function v on X representing $\succsim$. Let w be the restriction of v on $X^* \equiv \{\, \mathbf{x} \in X : \mathbf{x} = \lambda \mathbf{e}$ for some $\lambda > 0.\}$. Since w is monotone, w^{-1} is well-defined and w^{-1} is C^2. Setting v^* on X by $v^*(\mathbf{x}) = u \circ w^{-1} \circ v(\mathbf{x})$ for all $\mathbf{x} \in X$, it holds that v^* is C^2. Moreover, it holds by (A4) that v^* coincides with u on X, which implies that u is C^2. Since $\succsim$ is strict quasi-convex, it holds by

Friedman (1973) that the 1-homogeneous function u is weakly concave on X.

(**iii**) Let υ be a real-valued function X. If there exists $a > 0$ such that $v(\mathbf{x}) = a \cdot u(\mathbf{x})$ for all $\mathbf{x} \in X$, then it holds by Lemma 2(ii) that υ is 1-homogeneous and satisfies the condition: $\mathbf{x} \precsim \mathbf{y} \Leftrightarrow v(\mathbf{x}) \geq v(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$. If υ is 1-homogeneous and satisfies the condition: $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow v(\mathbf{x}) \geq v(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$. Since $v(\mathbf{x}) > 0$ for all $\mathbf{x} \in X$ by the 1-homogeneity and monotonicity of υ, setting $a = v(1, \cdots, 1) / u(1, \cdots, 1)$, we have that $a > 0$ and

$$v(t, \cdots, t) = tv(1, \cdots, 1) = t \cdot a \cdot u(1, \cdots, 1) = a \cdot u(t, \cdots, t) \quad \text{for all } t > 0. \tag{A5}$$

Fix any $\mathbf{x} \in X$. Since $\succsim$ is neo-classical, there is a positive number $t^0 > 0$ such that $(t^0, \cdots, t^0) \sim \mathbf{x}$, which implies that $u(\mathbf{x}) = u(t^0, \cdots, t^0)$ and $v(\mathbf{x}) = v(t^0, \cdots, t^0)$. Thus we have by (A5) that $u(\mathbf{x}) = u(t^0, \cdots, t^0) = a \cdot v(t^0, \cdots, t^0) = a \cdot v(\mathbf{x})$. □

Proof of Lemma 3: (**i**) For any $\mathbf{x} \in X$, since $\succsim$ is a continuous, monotone and strictly quasi-convex, there exists $\mathbf{p} \in P$ such that $\mathbf{x} \in D(\mathbf{p}, \mathbf{p} \cdot \mathbf{x})$, which implies $B \neq \emptyset$ and $P^* \neq \emptyset$. (**ii**) Suppose that $(\mathbf{p}, I) \in B$, which implies $D(\mathbf{p}, I) \neq \emptyset$. Since $\succsim$ is strictly convex, $D(\mathbf{p}, I)$ is a singleton. □

Proof of Lemma 4: Since $\succsim$ is smooth, i.e., $\succsim$ is represented by a C^2 utility function $u : X \to \mathbb{R}$ and the determinant of the bordered Hessian of $u(\mathbf{x})$ is non-zero for all $\mathbf{x} \in X$, it follows from Simon and Blume (1994, Theorem 22.2) that $d(\mathbf{p}, I)$ is C^1 for all $(\mathbf{p}, I) \in B$. □

Proof of Lemma 5: (**i**) Suppose that $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{c}*\mathbf{x} \succ \mathbf{c}*\mathbf{y}$ for some $\mathbf{c}, \mathbf{x}, \mathbf{y} \in X$. It holds by Claim 1 that

$$(\mathbf{x} \to \mathbf{x}) \sim_* (\mathbf{x} \to \mathbf{y}) \quad \text{and} \quad (\mathbf{c}*\mathbf{x} \to \mathbf{c}*\mathbf{x}) \succ_* (\mathbf{c}*\mathbf{x} \to \mathbf{c}*\mathbf{y}) \tag{A6}$$

It holds by strong homogeneity axiom that

$$(\mathbf{x} \to \mathbf{y}) \sim_* (\mathbf{c}*\mathbf{x} \to \mathbf{c}*\mathbf{y}) \tag{A7}$$

and

$$(\mathbf{x} \to \mathbf{x}) \sim_* (\mathbf{c}*\mathbf{x} \to \mathbf{c}*\mathbf{x}). \tag{A8}$$

Hence it holds by (A6) and (A7) that $(\mathbf{c}*\mathbf{x} \to \mathbf{c}*\mathbf{x}) \succ_* (\mathbf{c}*\mathbf{x} \to \mathbf{c}*\mathbf{y}) \sim_* (\mathbf{x} \to \mathbf{y}) \sim_* (\mathbf{x} \to \mathbf{x})$, which contradicts with (A8). Thus we have that $\mathbf{x} \sim \mathbf{y} \Rightarrow \mathbf{c}*\mathbf{x} \sim \mathbf{c}*\mathbf{y}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{c} \in X$.

(**ii**) It suffices to prove that $\mathbf{x} \succ \mathbf{y} \Rightarrow \mathbf{c}^**\mathbf{x} \succsim \mathbf{c}^**\mathbf{y}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{c}^* \in X$ such that $\mathbf{c}^* = (c, 1, \cdots, 1)$.

Suppose that $\mathbf{x} \succ \mathbf{y}$ and $\mathbf{c}^* * \mathbf{y} \succ \mathbf{c}^* * \mathbf{x}$ for some $\mathbf{c}^*, \mathbf{x}, \mathbf{y} \in X$.

Case 1($c > 1$): Since $\succsim$ is monotone and continuous on X, there is a continuous function u on X such that $\mathbf{x} \succsim \mathbf{y} \Leftrightarrow u(\mathbf{x}) \geq u(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$. Define $\mathbf{c}^*(t) = (t, 1, \cdots, 1)$ for all $t \in [1, c]$, and set $f(t) = u(\mathbf{c}^*(t) * \mathbf{x}) - u(\mathbf{c}^*(t) * \mathbf{y})$. Then it holds that $f(1) = u(\mathbf{x}) - u(\mathbf{y}) > 0$ and $f(c) = u(\mathbf{c}^* * \mathbf{x}) - u(\mathbf{c}^* * \mathbf{y}) < 0$. Since f is continuous on $[1, c]$, there exists some $t^* \in [1, c]$ such that $f(t^*) = u(\mathbf{c}^*(t^*) * \mathbf{x}) - u(\mathbf{c}^*(t^*) * \mathbf{y}) = 0$, which implies that $\mathbf{c}^*(t^*) * \mathbf{x} \sim \mathbf{c}^*(t^*) * \mathbf{y}$. It holds by Lemma 5(i) that $\mathbf{x} \sim \mathbf{y}$, which contradicts with $\mathbf{x} \succ \mathbf{y}$. Thus we have $\mathbf{x} \succ \mathbf{y} \Rightarrow \mathbf{c}^* * \mathbf{x} \succsim \mathbf{c}^* * \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{c}^* \in X$.

Case 2($1 > c > 0$): We can prove the assertion by almost the same manner in Case 1.

(**iii**) It holds by Trockel's (1989) theorem that there exists a Cobb-Douglas utility function representing $\succsim$. □

Proof of Lemma 6:(**i**) It holds by consistency axiom that $\mathbf{x} \sim \mathbf{x}^* \Rightarrow (\mathbf{x} \to \mathbf{y}^*) \sim_* (\mathbf{x}^* \to \mathbf{y}^*)$ and $\mathbf{y} \sim \mathbf{y}^* \Rightarrow (\mathbf{x} \to \mathbf{y}) \sim_* (\mathbf{x} \to \mathbf{y}^*)$. Hence we have that

$$(\mathbf{x} \to \mathbf{y}) \sim_* (\mathbf{x} \to \mathbf{y}^*) \sim_* (\mathbf{x}^* \to \mathbf{y}^*) . \qquad \text{(A9)}$$

Suppose that $\mathbf{x} \sim \mathbf{x}^*$, $\mathbf{y} \sim \mathbf{y}^*$, $\mathbf{z} \sim \mathbf{z}^*$ and $\mathbf{w} \sim \mathbf{w}^*$. Since it holds by (A9) that

$$\mathbf{x} \sim \mathbf{x}^* \text{ and } \mathbf{y} \sim \mathbf{y}^* \Rightarrow (\mathbf{x} \to \mathbf{y}) \sim_* (\mathbf{x}^* \to \mathbf{y}^*),$$

and since $\mathbf{z} \sim \mathbf{z}^*$ and $\mathbf{w} \sim \mathbf{w}^* \Rightarrow (\mathbf{z} \to \mathbf{w}) \sim_* (\mathbf{z}^* \to \mathbf{w}^*)$, if $(\mathbf{x} \to \mathbf{y}) \succsim_* (\mathbf{z} \to \mathbf{w})$, then $(\mathbf{x}^* \to \mathbf{y}^*) \sim_* (\mathbf{x} \to \mathbf{y}) \succsim_* (\mathbf{z} \to \mathbf{w})\ ) \sim_* (\mathbf{z}^* \to \mathbf{w}^*)$, which implies that $(\mathbf{x}^* \to \mathbf{y}^*) \succsim_* (\mathbf{z}^* \to \mathbf{w}^*)$. Similarly, we can prove that $(\mathbf{x}^* \to \mathbf{y}^*) \succsim_* (\mathbf{z}^* \to \mathbf{w}^*) \Rightarrow (\mathbf{x} \to \mathbf{y}) \succsim_* (\mathbf{z} \to \mathbf{w})$.

(**ii**) Claim 1(iii) implies Lemma 6(ii). □

Proof of Lemma 7: We have by Claim 1(iv) that there exists $a > 0$ and b such that

$$U(t\mathbf{e}) = aU^*(t\mathbf{e}) + b. \qquad \text{(A10)}$$

Fix any $\mathbf{x} \in X$. Since $\succsim$ is neo-classical, there is $I > 0$ such that $I\mathbf{e} \sim \mathbf{x}$, which implies that $U(I\mathbf{e}) = U(\mathbf{x})$ and $U^*(I\mathbf{e}) = U^*(\mathbf{x})$. Thus we have by this and (A10) that $U(\mathbf{x}) = U(I\mathbf{e}) = a \cdot U^*(I\mathbf{e}) + b = a \cdot U^*(\mathbf{x}) + b$. □

Proof of Lemma 8: Since $\log u(\mathbf{x})$ is a logarithmically homogeneous utility function, it holds that

(**i**) $d(\mathbf{p}, I) \succsim d(\mathbf{q}, J) \Leftrightarrow \log u(d(\mathbf{p}, I)) \geq \log u(d(\mathbf{q}, J)) \Leftrightarrow V^*(\mathbf{p}, I) \geq V^*(\mathbf{q}, J)$ for all $(\mathbf{p}, I), (\mathbf{q}, J) \in B$,

(**ii**) $[\ d(\mathbf{p}, I) \to d(\mathbf{q}, J)\ ] \succsim_* [\ d(\mathbf{p}^*, I^*) \to d(\mathbf{q}^*, J^*)]$

$\Leftrightarrow$ $\log u(d(\mathbf{q}, J)) - \log u(d(\mathbf{p}, I)) \geq \log u(d(\mathbf{q}^*, J^*)) - \log u(d(\mathbf{p}^*, I^*))$

$\Leftrightarrow$ $V^*(\mathbf{q}, J) - V^*(\mathbf{p}, I) \geq V^*(\mathbf{q}^*, J^*) - V^*(\mathbf{p}^*, I^*)$ for all $(\mathbf{p}, I), (\mathbf{q}, J), (\mathbf{p}^*, I^*), (\mathbf{q}^*, J^*) \in B$,

which means that V^* is an indirect utility function of $(\succsim, \succsim_*)$. □

Proof of Lemma 9: (**A**) By the definitions of B and P^*, it holds that $B \subset P^* \times \mathbb{R}_{++}$. We will prove $P^* \times \mathbb{R}_{++} \subset B$. Suppose that $\mathbf{p} \in P^*$, which implies $d(\mathbf{p}, I^*) \in X$ for some $I^* > 0$. It holds by Lemma 2(i) that $(I/I^*) \cdot d(\mathbf{p}, I^*) = d(\mathbf{p}, I) \in X$ for all $I > 0$. Hence we have that $P^* \times \mathbb{R}_{++} \subset B$. (**B**) (**i**) Suppose that $(\mathbf{p}, I) \in B$. It holds by lemma 2(i) and Lemma 9(A) that $(1/\lambda) \cdot d(\mathbf{p}, I) = d(\lambda\mathbf{p}, I) \in X$ for all $\lambda > 0$, which implies $(\lambda\mathbf{p}, I) \in B$ for all $\lambda > 0$. (**ii**) Suppose that $(\mathbf{p}, I) \in B$. It holds by lemma 2(i) and Lemma 9(A) that $(\mathbf{p}, \lambda I) \in B$ and $\lambda \cdot d(\mathbf{p}, I) = d(\mathbf{p}, \lambda I) \in X$ for all $\lambda > 0$. □

Proof of Lemma 10: Suppose that V is an indirect utility function of $(\succsim, \succsim_*)$. Define $\mathbf{p^*} \in P^*$ by $\mathbf{p^*} = (\partial u(\mathbf{e})/\partial x_1, \partial u(\mathbf{e})/\partial x_2, \cdots, \partial u(\mathbf{e})/\partial x_n)$. It holds by Lemma 9(B) that there exists unique $t^* > 0$ such that $t^* \cdot d(\mathbf{p^*}, 1) = \mathbf{e}$. Define $\mathbf{p}^0 \in P^*$ by $\mathbf{p}^0 = (1/t^*)\mathbf{p^*}$. Then we have that

$$d(\mathbf{p}^0, 1) = d((1/t^*)\mathbf{p^*}, 1) = t^* \cdot d(\mathbf{p^*}, 1) = \mathbf{e}.$$

Define two functions f and g on $\mathbb{R}_{++}$ by $f(t) = V(\mathbf{p}^0, t)$ and $g(t) = V^*(\mathbf{p}^0, t)$. It holds that

(**i**) $f(t) \geq f(s) \Leftrightarrow V(\mathbf{p}^0, t) \geq V(\mathbf{p}^0, s) \Leftrightarrow d(\mathbf{p}^0, t) \succsim d(\mathbf{p}^0, s) \Leftrightarrow td(\mathbf{p}^0, 1) \succsim sd(\mathbf{p}^0, 1)$

$\Leftrightarrow t \cdot \mathbf{e} \succsim s \cdot \mathbf{e}$ for all $t, s \in \mathbb{R}_{++}$

(**ii**) $f(s) - f(t) \geq f(w) - f(z) \Leftrightarrow V(\mathbf{p}^0, s) - V(\mathbf{p}^0, t) \geq V(\mathbf{p}^0, w) - V(\mathbf{p}^0, z)$

$\Leftrightarrow (t \cdot \mathbf{e} \rightarrow s \cdot \mathbf{e}) \succsim_* (z \cdot \mathbf{e} \rightarrow w \cdot \mathbf{e})$ for all $t, s, z, w \in \mathbb{R}_{++}$.

We have by Claim 1(iv) that there exists $a > 0$ and b such that

$$V(\mathbf{p}^0, I) = a \cdot V^*(\mathbf{p}^0, I) + b \text{ for all } I \in \mathbb{R}_{++}. \quad \text{(A11)}$$

Fix any $(\mathbf{p}, I) \in B$. Since $\succsim$ is neo-classical, there is $I^0 > 0$ such that $d(\mathbf{p}^0, I^0) \sim d(\mathbf{p}, I)$, which implies that $V(\mathbf{p}^0, I^0) = V(\mathbf{p}, I)$ and $V^*(\mathbf{p}^0, I^0) = V^*(\mathbf{p}, I)$. Thus we have by this and (A11) that $V(\mathbf{p}, I) = V(\mathbf{p}^0, I^0) = a \cdot V^*(\mathbf{p}^0, I^0) + b = a \cdot V^*(\mathbf{p}, I) + b$. □

Appendix B

Proof Claim 1: (**i**) It holds by the consistency axiom that

$$\mathbf{x} \sim \mathbf{y} \Rightarrow \mathbf{x} \succsim \mathbf{y} \text{ and } \mathbf{y} \succsim \mathbf{x} \Rightarrow (\mathbf{z} \rightarrow \mathbf{x}) \succsim_* (\mathbf{z} \rightarrow \mathbf{y}) \text{ and } (\mathbf{z} \rightarrow \mathbf{y}) \succsim_* (\mathbf{z} \rightarrow \mathbf{x})$$

$$\Rightarrow\ (\mathbf{z}\to\mathbf{x}) \sim_* (\mathbf{z}\to\mathbf{y}) \quad \text{for all } \mathbf{z}\in X.$$

(**ii**) It holds by the contraposition of the consistency axiom that

$$\textbf{not } \mathbf{y} \succsim \mathbf{x} \ \Rightarrow\ \textbf{not } (\mathbf{z}\to\mathbf{y}) \succsim_* (\mathbf{z}\to\mathbf{x}).$$

It holds by the consistency axiom and this that

$$\mathbf{x} \succ \mathbf{y} \ \Rightarrow\ \mathbf{x} \succsim \mathbf{y} \text{ and } \textbf{not } \mathbf{y} \succsim \mathbf{x} \ \Rightarrow\ (\mathbf{z}\to\mathbf{x}) \succsim_* (\mathbf{z}\to\mathbf{y}) \text{ and } \textbf{not } (\mathbf{z}\to\mathbf{y}) \succsim_* (\mathbf{z}\to\mathbf{x})$$

$$\Rightarrow\ (\mathbf{z}\to\mathbf{x}) \succ_* (\mathbf{z}\to\mathbf{y}) \quad \text{for all } \mathbf{z}\in X.$$

(**iii**) We need a claim proved by Miyake (2014, Proposition):

Claim 2: Let $(\succsim^0, \succsim^0_*)$ be a preference on $\mathbb{R}_{++}$, and suppose that $(\succsim^0, \succsim^0_*)$ satisfies the following conditions:

(**a**) $t > s \Rightarrow t \succ^0 s$ for all $t, s \in \mathbb{R}_{++}$.

(**b**) There exists some $z \in \mathbb{R}_{++}$ such that $(z \to x) \succsim^0_* (z \to y) \Leftrightarrow x \succsim^0 y$ for all $x, y \in \mathbb{R}_{++}$.

(**c**) $(x \to y) \sim^0_* (tx \to ty)$ for all $x, y \in \mathbb{R}_{++}$ and all $t > 0$.

Then the following assertions hold:

(**i**) $\log t \geq \log s \Leftrightarrow t \succsim^0 s$ for all $t, s \in \mathbb{R}_{++}$

(**ii**) $\log s - \log t \geq \log w - \log z \Leftrightarrow (t \to s) \succsim^0{}_* (z \to w)$ for all $t, s, z, w \in \mathbb{R}_{++}$.

(**iii**) If f is a real-valued function on $\mathbb{R}_{++}$ satisfying the two conditions: $f(t) \geq f(s) \Leftrightarrow t \succsim^0 s$ for all $t, s \in \mathbb{R}_{++}$; $f(s) - f(t) \geq f(w) - f(z) \Leftrightarrow (t \to s) \succsim^0{}_* (z \to w)$ for all $t, s, z, w \in \mathbb{R}_{++}$, then there exists $a > 0$ and b such that $f(t) = a\cdot\log t + b$ for all $t \in \mathbb{R}_{++}$.

Define a preference $(\succsim^1, \succsim^1_*)$ on $\mathbb{R}_{++}$ by

$$t \succsim^1 s \ \Leftrightarrow\ t\cdot\mathbf{e} \succsim s\cdot\mathbf{e} \ \text{ for all } t, s \in \mathbb{R}_{++}; \tag{A12}$$

$$(t \to s) \succsim^1_* (z \to w) \ \Leftrightarrow\ (t\cdot\mathbf{e} \to s\cdot\mathbf{e}) \succsim_* (z\cdot\mathbf{e} \to w\cdot\mathbf{e}) \ \text{ for all } t, s, z, w \in \mathbb{R}_{++}, \tag{A13}$$

Since $(\succsim, \succsim_*)$ on X satisfies the consistency and homogeneity axioms, $(\succsim^1, \succsim^1_*)$ on $\mathbb{R}_{++}$ satisfies the conditions Claim 2(a, b, c). Hence it holds Claim 2(ii) that

$$\log s - \log t \geq \log w - \log z \ \Leftrightarrow\ (t \to s) \succsim^1{}_* (z \to w) \quad \text{for all } t, s, z, w \in \mathbb{R}_{++}.$$

Hence we have by (A13) that

$$\log s - \log t \geq \log w - \log z \ \Leftrightarrow\ (t\cdot\mathbf{e} \to s\cdot\mathbf{e}) \succsim_* (z\cdot\mathbf{e} \to w\cdot\mathbf{e}) \quad \text{for all } t, s, z, w \in \mathbb{R}_{++}.$$

(**iv**) Suppose that f is a real-valued function on $\mathbb{R}_{++}$ satisfying the two conditions:

$$f(t) \geq f(s) \Leftrightarrow t\cdot\mathbf{e} \succsim s\cdot\mathbf{e} \text{ for all } t, s \in \mathbb{R}_{++};$$

$$f(s) - f(t) \geq f(w) - f(z) \Leftrightarrow (t\cdot\mathbf{e} \to s\cdot\mathbf{e}) \succsim_{*} (z\cdot\mathbf{e} \to w\cdot\mathbf{e}) \text{ for all } t, s, z, w \in \mathbb{R}_{++}.$$

It holds by (A12) and (A13) that

$$f(t) \geq f(s) \Leftrightarrow t \succsim^{1} s \text{ for all } t, s \in \mathbb{R}_{++};$$

$$f(s) - f(t) \geq f(w) - f(z) \Leftrightarrow (t \to s) \succsim^{1}{}_{*} (z \to w) \text{ for all } t, s, z, w \in \mathbb{R}_{++}.$$

We have by Claim 2(iii) that there exists $\alpha > 0$ and β such that

$$f(t) = \alpha\cdot\log t + \beta \quad \text{for all } t \in \mathbb{R}_{++}. \qquad \text{(A14)}$$

Using the almost the same arguments, we can show that there exists $\alpha^* > 0$ and β^* such that

$$g(t) = \alpha^*\cdot\log t + \beta^* \quad \text{for all } t \in \mathbb{R}_{++}, . \qquad \text{(A15)}$$

Hence we have by (A14) and (A15) that $[\, U(t) - \beta\,]/\alpha = \log t = [\, U^*(t) - \beta^*\,]/\alpha^*$, which implies that

$$f(t) = \alpha[\, U^*(t) - \beta^*\,]/a^* + \beta = (\alpha/\alpha^*)\, g(t) - (\alpha\beta^*)/\alpha^* + \beta.$$

Setting $a = (\alpha/\alpha^*)$ and $b = -(\alpha\beta^*)/\alpha^* + \beta$, we have that $f(t) = ag(t) + b$. □

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