Local Independence, Monotonicity and Axiomatic Characterization of Price-Money Message Mechanism*†

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September 11, 2015

Abstract

To characterize money in static economic model, it is known to be important to consider the agent-commodity double infinity settings, i.e., the overlapping-generations framework. There does not seem to exist any papers, however, treating the axiomatic characterization problems for such monetary Walras allocations under the social choice and/or mechanism design settings. We show that the monetary Walras allocation for the economy with double infinities is characterized through the *weak Pareto-optimality*, the *individual rationality*, the *local independence* and/or the *monotonicity* conditions of the *social choice correspondence* among the *message mechanisms* under the *category theoretic* approach in Sonnenschein (1974). We utilize Sonnenschein's market extension axiom for *swamped* economies that is closely related to the *replica stability* axiom of Thomson (1988). We can see how these conditions characterize the price-money message mechanism *universally* among a wide class of mechanisms, and *efficiently* in the sense that it has the minimal message spaces (*price-money dictionary theorems*). Moreover, by using the category theoretic framework, we can obtain the up-to-isomorphism uniqueness for such a dictionary object (*isomorphism theorems*).

Keywords : Message Mechanism, Social Choice Correspondence, Overlapping-Generations Economy, Monetary Walras Allocation, Local Independence, Monotonicity, Informational Efficiency, Universal Mapping Property

JEL Classification: C60, D50, D51, D71, E00

*We wish to express special gratitude to professor Ryoichi Nagahisa (Kansai University) for his important comments and advice. We are also grateful for comments by professors Tomoyuki Kamo (Kyoto Sangyo University) and Kazuhiko Hashimoto (Osaka University), and to participants in the seminar at Osaka University (August 7, 2015). A part of this research was supported by the Grant-in-Aid for Scientific Research from the Japan Society for the Promotion of Science in 2013 (No.25380227) and the Grant-in-Aid for JSPS Fellows of the Japan Society for the Promotion of Science in 2015 (No.15J01034).

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1 Introduction

To introduce *money* in a static general equilibrium model, the overlapping-generations model with the double infinity of commodities and agents is known to be the most fundamental framework. The model is firstly introduced by Samuelson in 1958 (Samuelson 1958), and brought about various discussions because of its outstanding feature that *competitive equilibria may not necessarily be Pareto-optimal.* Although Samuelson characterized the role of fiat money as a certain kind of *social contract* to lead Pareto improvement, his argument was not accurate enough. In 1970s and 80s, many papers on monetary general equilibria make certain that the existence of fiat money may not necessarily cause Pareto improvements nor necessarily assure the existence of monetary equilibria that may or may not be Pareto-optimal (see, for example, Shell 1971, Hayashi 1976 and Okuno and Zilcha 1980). Moreover, a monetary equilibrium is unstable from the cooperative game theoretic viewpoint (Esteban 1986) as well as from the viewpoint of equilibrium dynamics (Gale 1973). The only affirmative result that we have for a characterization of equilibrium in overlapping-generations economies with money is the relation between *weakly Pareto-optimality* and valuation equilibrium (see Balasko and Shell 1980 and Esteban (1986)).

Recently, the authors presented a replica finite core equivalence and characterization for the monetary Walrasian correspondence under the overlapping-generations framework and provided a axiomatic characterization for it.[1] Except for our works, however, there does not seem to exist papers treating such axiomatic characterization problems for the double infinity monetary equilibrium allocations under the social choice and mechanism design settings. In this paper, we treat this problem through the setting of the *message mechanism* like Sonnenschein (1974), Hurwicz (1960), Mount and Reiter (1974), Osana (1978) and Jordan (1982). We especially follow the approach in Sonnenschein (1974) and Sonnenschein's market extension axiom for *swamped* economies which is closely related to the *replica stability* axiom of Thomson (1988) and Nagahisa (1994) for the social choice framework. We show that the monetary equilibrium allocation and the price-money message mechanism are possible to be characterized axiomatically by using well-known social-choice theoretic normative criteria, especially, the local independency and the monotonicity through the category theoretic approach of Sonnenschein (1974).

Theorem 1 shows in the domain of double infinity exchange economies, sufficient and/or necessary relations between the social choice correspondence satisfying the property of *weak Pareto-optimality* (WPO), *individual rationality* (IR), the *local independence* (LI) and the correspondence that allocate for each economy its monetary Walrasian equilibria, the *monetary Walrasian correspondence.* Based on this knowledge, Theorem 2 asserts that every *message mechanism* in the sense of Sonnenschein (1974) whose equilibrium results are compatible with WPO, IR and LI allocations can *universally* and *uniquely* be identified with a part of monetary Walrasian (price-money) message mechanism. We can also obtain Theorem 3 that assures the *uniqueness* of such a message space as a solution to the *universal mapping problem.* Theorem 4 and 5 show that the price-money dictionary theorem (Theorem 2) and the isomorphism theorem (Theorem 3) can also be obtained through the *monotonicity* instead of the *local independency.*

2 The Model

As in our previous paper (Urai and Murakami 2015b), we define the general overlapping-generations settings under the duality between $\boldsymbol{R}_\infty$ and $\boldsymbol{R}^\infty$. Since we are concerned with one shot (perfect foresight)

[1] See Urai and Murakami (2015a) and Urai and Murakami (2015b).

equilibrium states, this kind of overlapping-generations model (one good for each period and $(\ell(t)+1)$-periods lifetime-span for each generation t) is sufficiently general to include all types with ℓ-goods and n-periods lifetime for each generation t.[2]

We denote by $\boldsymbol{N}$ the set of all positive integers and by $\boldsymbol{R}$ the set of real numbers. An *overlapping-generations economy*, or an *economy*, $\mathcal{E}$, is a list of:

(OG1) $\{I_t\}_{t=1}^{\infty}$; a countable family of mutually disjoint finite subsets of $\boldsymbol{N}$ such that $\bigcup_{t=1}^{\infty} I_t = \boldsymbol{N}$, where $I_t \neq \emptyset$ for each $t \in \boldsymbol{N}$. I_t is the index set of agents in *generation* t.

(OG2) $\{K_t\}_{t=1}^{\infty}$; a countable family of non-empty finite intervals, $K_t = \{k(t), k(t)+1, \cdots, k(t)+\ell(t)\}$ where $k(t)$ and $\ell(t)$ are elements of $\boldsymbol{N}$ such that $\bigcup_{t=1}^{\infty} K_t = \boldsymbol{N}$ and $k(t) \leqq k(t+1) \leqq k(t)+\ell(t)$ for all $t \in \boldsymbol{N}$, and $\{t \mid n \in K_t\}$ is finite for each $n \in \boldsymbol{N}$. K_t is the index set of *commodities* available to generation t.

(OG3) $\{(\precsim_i,\ \omega_i)\}_{i \in \bigcup_{t\in \boldsymbol{N}} I_t}$; countably many agents, where $\precsim_i$ is a rational weak preference on the commodity space, $\boldsymbol{R}^{K_t}$, of $i \in I_t$ for $t \in \boldsymbol{N}$. Every preference, $\precsim_i$, can be represented by a continuous utility function, $u_i : \boldsymbol{R}^{K_t} \to \boldsymbol{R}$, that is strictly quasi-concave and strictly monotonic. The *initial endowment* of i, ω_i, is an element of $\boldsymbol{R}_{++}^{K_t} = \{x \mid x : K_t \to \boldsymbol{R}_{++}\}$ for each $i \in I_t$.

The commodity space for each generation, $\boldsymbol{R}_{+}^{K_t}$, can be recognized as the subset of $\boldsymbol{R}^{\boldsymbol{N}}$, the set of all functions from $\boldsymbol{N}$ to $\boldsymbol{R}$, by identifying $x \in \boldsymbol{R}_{+}^{K_t}$ with the function that takes value 0 on $\boldsymbol{N}\backslash K_t$. The total commodity space for an economy is, therefore, the set of all finite sums in $\boldsymbol{R}^{\boldsymbol{N}}$ among points in commodity spaces of some generations, $\oplus_{t=1}^{\infty}\boldsymbol{R}_{+}^{K_t} \subset \boldsymbol{R}^{\boldsymbol{N}}$. Clearly, $\oplus_{t=1}^{\infty}\boldsymbol{R}_{+}^{K_t}$ can also be identified with a subset of the direct sum, $\boldsymbol{R}_{\infty}$, the set of all finite real sequences, which is a subspace of the set of all real sequences, $\boldsymbol{R}^{\infty} \approx \boldsymbol{R}^{\boldsymbol{N}}$.

Given an economy, $\mathcal{E} = (\{I_t\}_{t=1}^{\infty},\ \{K_t\}_{t=1}^{\infty},\ \{(\precsim_i,\ \omega_i)\}_{i \in \bigcup_{t\in \boldsymbol{N}} I_t})$, the *price space* for $\mathcal{E}$, $\mathcal{P}(\mathcal{E})$, is defined as the set of all p in $\boldsymbol{R}_{+}^{\boldsymbol{N}}$ such that under the duality between $\boldsymbol{R}_{\infty}$ and $\boldsymbol{R}^{\infty}$, p evaluates all agents' initial endowments positively, i.e.,

$$\mathcal{P}(\mathcal{E}) = \{p \in \boldsymbol{R}_{+}^{\boldsymbol{N}} \mid p \cdot \omega_i > 0 \text{ for all } i \in I_t, \text{ for all } t \in \boldsymbol{N}\}. \tag{1}$$

Since for all $i \in I_t$, $\omega_i \in \boldsymbol{R}_{++}^{K_t}$, for all $t \in \boldsymbol{N}$, the price space of $\mathcal{E}$ always includes $\boldsymbol{R}_{++}^{\boldsymbol{N}}$ for all $\mathcal{E}$ in $\boldsymbol{\mathcal{E}con}$, the set of all economies satisfying conditions (OG1), (OG2) and (OG3).

For each $\mathcal{E} = (\{I_t\},\ \{K_t\},\ \{(\precsim_i,\ \omega_i)\}) \in \boldsymbol{\mathcal{E}con}$, sequence $(x_i \in \boldsymbol{R}^{K_t})_{i \in \bigcup_{t\in \boldsymbol{N}} I_t}$ is called an *allocation* for $\mathcal{E}$. An allocation $(x_i \in \boldsymbol{R}^{K_t})_{i \in \bigcup_{t\in \boldsymbol{N}} I_t}$ is said to be *feasible* if

$$\sum_{t\in \boldsymbol{N}}\sum_{i \in I_t} x_i \leqq \sum_{t\in \boldsymbol{N}}\sum_{i \in I_t} \omega_i, \tag{2}$$

where the summability in $\boldsymbol{R}^{\boldsymbol{N}}$ of both sides of the equation is assured by (OG2). The list of a price vector $p^* \in \mathcal{P}(\mathcal{E})$, a *non-negative wealth transfer function* $M_{\mathcal{E}}^* : \boldsymbol{N} = \bigcup_{t=1}^{\infty} I_t \to \boldsymbol{R}_{+}$, and a feasible allocation $(x_i^* \in \boldsymbol{R}^{K_t})_{i \in \bigcup_{t\in \boldsymbol{N}} I_t}$ is called a *monetary Walras allocation* for $\mathcal{E}$, if for each $t \in \boldsymbol{N}$ and $i \in I_t$, x_i^* is a $\precsim_i$-greatest element in the set $\{x_i \in \boldsymbol{R}^{K_t} \mid p^* \cdot x_i \leqq p^* \cdot \omega_i + M_{\mathcal{E}}^*(i)\}$ (see, e.g., Balasko and Shell 1981 and Esteban and Millán 1990). We denote the set of all monetary Walras allocations by $\boldsymbol{\mathcal{MW}alras}(\mathcal{E})$.

[2] With respect to the topological structure for this kind of double infinity economies, we also use the simplest case of the duality between the direct limit of finite dimensional commodity spaces and the inverse limit of finite dimensional price spaces (see Aliprantis et al. 1989, Urai 1990 and Urai 1994), which is equivalent to treat the duality between $\boldsymbol{R}_{\infty}(\subset \boldsymbol{R}^{\infty})$ and $\boldsymbol{R}^{\infty}$ under the ordinary product topology.

An allocation, x, for economy $\mathcal{E} = (\{I_t\},\ \{K_t\},\ \{(\succsim_i,\ \omega_i)\}) \in \mathcal{E}con$ is said to be *weakly Pareto-optimal* (WPO), if there is no y with the property $\sum_{t\in \boldsymbol{N}} \sum_{i\in I_t} y_i = \sum_{t\in \boldsymbol{N}} \sum_{i\in I_t} x_i$, $y_i = x_i$ except for a finite number of i, and $y_i \succsim_i x_i$ with at least one strict preference $\succ_i$ for $i \in \bigcup_{t\in \boldsymbol{N}} I_t$.[3] Moreover, we say that allocation x is *individually rational* (IR) if $x_i \succsim_i \omega_i$ for all $i \in \bigcup_{t\in \boldsymbol{N}} I_t$.

Social choice correspondence g on domain $\mathcal{E}con$ is a correspondence that assigns some allocations for each $\mathcal{E}$, $g : \mathcal{E}con \ni \mathcal{E} \mapsto g(\mathcal{E}) \subset \boldsymbol{R}^{\boldsymbol{N}}$. Let $K(s) = \bigcup_{t=1}^{s} K_t$ and $I(s) = \bigcup_{t=1}^{s} I_t$ for each $s \in \boldsymbol{N}$. We say that social choice correspondence g satisfies the condition of *local independence* (LI) if $x \in g(\mathcal{E})$ implies $x \in g(\mathcal{E}')$ whenever $\mathcal{E}$ is $(\{I_t\},\ \{K_t\},\ \{(\succsim_i,\ \omega_i)\})$, $\mathcal{E}'$ is $(\{I_t\},\ \{K_t\},\ \{(\succsim_i',\ \omega_i)\})$, $\succsim_i' = \succsim_i$ except for finite agents, and there exists $s \in \boldsymbol{N}$ such that $I(s)$ includes all such finite agents and a unique supporting hyperplane $H_s \subset \boldsymbol{R}^{K(s)}$ at $x \in g(\mathcal{E})$ of every better set $\{y_i |\, y_i \succ_i x_i\}$ of $i \in I(s)$ also supports every $\{y_i |\, y_i \succ_i' x_i\}$ of $i \in I(s)$.[4] Social choice correspondence g is said to satisfy the condition of *monotonicity* if $x \in g(\mathcal{E})$ implies $x \in g(\mathcal{E}')$ whenever $\mathcal{E}$ is $(\{I_t\},\ \{K_t\},\ \{(\succsim_i,\ \omega_i)\})$, $\mathcal{E}'$ is $(\{I_t\},\ \{K_t\},\ \{(\succsim_i',\ \omega_i)\})$, $\succsim_i' = \succsim_i$ except for finite agents, and there exists $s \in \boldsymbol{N}$ such that $I(s)$ includes all such finite agents and every better set $\{y_i |\, y_i \succ_i x_i\}$ at x of $i \in I(s)$ includes $\{y_i |\, y_i \succ_i' x_i\}$ of each $i \in I(s)$.

3 A Preliminary Theorem

Social choice correspondence $g : \mathcal{E}con \to \boldsymbol{R}^{\boldsymbol{N}}$ is said to be *monetary Walrasian* if $g(\mathcal{E}) = \mathcal{MW}alras(\mathcal{E})$. We have the next theorem that characterizes the monetary Walrasian social choice correspondence through the LI condition.

Theorem 1 (LI Characterization Theorem) : (i) Assume that social choice correspondence g is WPO and IR. If g satisfies LI and all better sets of $i \in I(s)$ at $x \in g(\mathcal{E})$ is supported by a unique price p in $\boldsymbol{R}^{K(s)}$, then for all $i \in I(s)$, their initial endowments are evaluated less than or equal to the value of x_i under p. Especially, if g satisfies LI and all better sets of agents at $x \in g(\mathcal{E})$ is supported by a unique price, x is a monetary Walras allocation (g is monetary Walrasian). (ii) On the other hand, monerary Walrasian social choice correspondence g is WPO and IR valued, and if the monetary message $M_{\mathcal{E}}$ does not depend in $(\succsim_i)_{i\in\bigcap_{i\in \boldsymbol{N}} I_t}$, it satisfies the LI condition.

Proof: Assume that an allocation $x \in g(\mathcal{E})$ is WPO and IR, and social choice correspondence g satisfies the LI. By the weak Pareto-optimality, we have a price, $p \in \boldsymbol{R}^{\infty}$, such that for each $i \in \bigcup_{t=1}^{\infty} I_t$, $x_i' \succ_i x_i$ implies that $p \cdot x_i' > p \cdot x_i$ (Balasko and Shell 1980). If $p \cdot x_i \geqq p \cdot \omega_i$ for all i, by defining $M_{\mathcal{E}}(i)$ as $p \cdot x_i - p \cdot \omega_i \in \boldsymbol{R}_+$, we can identify x as a monetary Walras allocation. Hence, the latter part of the assertion (i) follows from the former. Suppose that for some $i \in I(s)$, $p \cdot x_i < p \cdot \omega_i$ and all better sets of $i \in I(s)$ at x is supported by a unique price $p \in \boldsymbol{R}^{K(s)}$. Then, it is possible to change the preference of i to $\succ_i'$ so that $\succ_i'$ satisfies $\omega_i \succ_i' x_i$ and all assumptions in (OG3), and p remains to be a supporting hyperplane of the better set of i under $\succ_i'$ at x. Under the LI, x should be in the value of the social choice for such an economy, which is impossible, however, since $\omega_i \succ_i' x_i$ contradicts to the IR.

On the other hand, monetary Walras allocation is obviously IR and is well known to be WPO (see Balasko and Shell 1980 and Esteban 1986). Moreover, for each allocation $x \in g(\mathcal{E})$ having a unique supporting hyperplane $H_s \in \boldsymbol{R}^{K(s)}$ for better sets of agents in $I(s)$ for some $s \in \boldsymbol{N}$, we can confirm that

[3] See Balasko and Shell (1980).
[4] See Nagahisa (1991).

the monetary Walrasian social choice correspondence g satisfies the LI. Indeed, as long as each non-negative wealth transfer message does not depend on preferences, any preference changes from economy $\mathcal{E}$ to $\mathcal{E}'$ to check the LI condition do not affect the property of allocation x to be agents' individual price-wealth maximands. ■

Note that in the definition of LI, the uniqueness property of the supporting hyperplane for better sets of agents at allocation x of g is important. The LI condition does not say anything for allocations that do not have this uniqueness property. For differentiable class of economies (treated in section 4 Theorem 3), the uniqueness property is satisfied at every allocations.

4 Axiomatic Characterization of the Message Mechanism

Three conditions in Theorem 1, the weak Pareto-optimality (WPO), the individual rationality (IR), and the local independence (LI), enable us to provide an axiomatic characterization of the monetary Walrasian message mechanisms through the category theoretic framework as in Sonnenschein (1974). We formulate, especially, the local independence condition and, interchangeably, the monotonicity condition as axioms for such message mechanisms.

At first, we reformulate the concepts in Sonnenschein (1974) into the social choice settings. A *WPO-IR compatible social choice correspondence* associates with each economy $\mathcal{E}$ a set of allocations which are WPO and IR allocations for $\mathcal{E}$. A *private representation*[5] of such a social choice correspondence, g, is a triple, (A, μ, f): the set A is a *message domain*, μ is a correspondence which indicates for each economy $\mathcal{E}$ the set $\mu(\mathcal{E}) \subset A$ of *equilibrium messages for* $\mathcal{E}$, and f is a function which defines for each agent and each message the *response* of the agent to the message. The *list of equilibrium responses* associated with $a \in \mu(\mathcal{E})$ assigns to each agent in $\mathcal{E}$ his response to the message a. (A, μ, f) is said to be a private representation of the social choice correspondence g if for each economy $\mathcal{E}$, $g(\mathcal{E})$ is a set of equilibrium lists of responses associated with the messages in $\mu(\mathcal{E})$.

The monetary Walrasian social choice correspondence associates with each economy the monetary Walras allocations of the economy. The standard private representation is such that $A = \boldsymbol{R}_+^{\boldsymbol{N}} \times \{M \mid M : \boldsymbol{\mathcal{E}con} \to \boldsymbol{R}_+^{\boldsymbol{N}}\}$, $\mu(\mathcal{E})$ is the set of equilibrium prices with non-negative wealth transfer of $\mathcal{E}$, and f gives the excess demand function of each consumer relative to price-money messages. Let us consider the following axioms.

Axiom S (Sonnenschein): For each finite list of the economies and the members, $(i_1, \mathcal{E}_1)$, $(i_2, \mathcal{E}_2)$, $\dots$, $(i_m, \mathcal{E}_m)$, each message $a \in A$ and each list of responses $(f_{i_s}(\mathcal{E}_s, a))_{s=1}^m$, there exists an economy $\mathcal{E}_*$ including $\{i_1, i_2, \dots, i_m\}$ such that a is an equilibrium message for $\mathcal{E}_*$ satisfying that the equilibrium list, $(f_i(\mathcal{E}_*, a))_{i=1}^\infty$, is an extension of $(f_{i_s}(\mathcal{E}_s, a))_{s=1}^m$.

The above condition is closely related to the replica stability axiom of Thomson (1988) (see Urai and Murakami 2015b). Note that since the non-negative wealth transfer may be different among agents having the same individual characteristics, it would be desirable to treat general messages that are *partly* economy-dependent.[6] Hence the finite agents in the previous axioms should be listed with the economies to which they belong.

[5] We use the word "private" in the sense that the responses are described in the form of private actions even though the messages are partly dependent on the economy.

[6] In the sense that the message is related not only to each agent's characteristics, i.e., the initial endowment and the preference, but also to their places in the economy to which they belong.

Next axioms redefines the local independency and the monotonicity conditions in the previous section through the terms in the message mechanism.

Axiom L (Local Independency) : For each economy $\boldsymbol{\mathcal{E}}$ and message a, if there exist generation $s \in \boldsymbol{N}$ and a unique hyperplane $H_s \subset \boldsymbol{R}^{K(s)}$ that supports the better set at $f_i(\boldsymbol{\mathcal{E}}, a)$ of every $i \in I(s)$, then for each economy $\boldsymbol{\mathcal{E}}'$ having the same indices of agents and commodities, endowments and possibly different preferences of agents in $I(s)$ of economy $\boldsymbol{\mathcal{E}}$, such that H_s is also a supporting hyperplane of the better set at $f_i(\boldsymbol{\mathcal{E}}, a)$ of every $i \in I(s)$, we have $f(\boldsymbol{\mathcal{E}}, a) = f(\boldsymbol{\mathcal{E}}', a)$.[7]

Axiom M (Monotonicity) : For each economy $\boldsymbol{\mathcal{E}}$ and message a, if $s \in \boldsymbol{N}$ and $\boldsymbol{\mathcal{E}}'$ is an economy having the same indices and endowments of agents in $\boldsymbol{\mathcal{E}}$ together with the same preferences except for agents in $I(s)$ such that every better set at $f_i(\boldsymbol{\mathcal{E}}, a)$ in $\boldsymbol{\mathcal{E}}$ includes the better set at the same point in $\boldsymbol{\mathcal{E}}'$ for each $i \in I(s)$, then we have $f(\boldsymbol{\mathcal{E}}, a) = f(\boldsymbol{\mathcal{E}}', a)$.

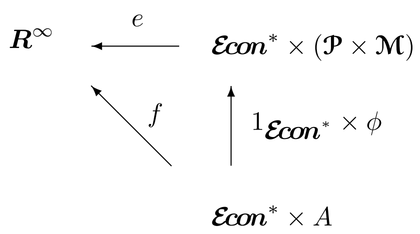
Consumer i is a pair $(\succsim_i, \omega_i)$, where $\succsim_i$ and ω_i satisfy the conditions in (OG3). We assume in the following the commodity structure, $\{K_t\}_{t=1}^{\infty}$, is fixed, and identify the set of all economies, $\boldsymbol{\mathcal{E}con}^*$, with the set of those in $\boldsymbol{\mathcal{E}con}$ with the commodity structure $\{K_t\}_{t=1}^{\infty}$. Denote by $I(t)$ the set of all agents in generations from 1 to t, i.e., $I(t) = \bigcup_{s=1}^{t} I_s$, and by $K(t)$ the set of all commodities that are available for agents in $I(t)$, i.e., $K(t) = \bigcup_{s=1}^{t} K_s$. For each t, by $\Delta^{K(t)}$, we denote the unit simplex in $\boldsymbol{R}^{K(t)}$ and by $\Delta_{++}^{K(t)}$ its relative interior, $\boldsymbol{R}_{++}^{K(t)} \cap \Delta^{K(t)}$. Let us consider projective system $(\Delta_{++}^{K(t')}, \varrho_{t't})_{t',t \in \boldsymbol{N}}$ and projective limit $\Delta_{++} = \varprojlim (\Delta_{++}^{K(t')}, \varrho_{t't})$, where $\varrho_{t't} : \Delta_{++}^{K(t)} \to \Delta_{++}^{K(t')}$ is defined as $\varrho_{t't}(p) = \frac{\mathrm{pr}_{K(t')}\, p}{\| \mathrm{pr}_{K(t')}\, p \|}$. Note that Δ_{++} can be recognized as a subset of $\boldsymbol{R}_{++}^{\infty}$ by identifying the equivalence class $[(x^t)_{t=1}^{\infty}]$ of $(x^t)_{t=1}^{\infty} \in \prod_{t=1}^{\infty} \Delta_{++}^{K(t)}$ with the element $p \in \boldsymbol{R}_{++}^{\infty}$ such that $\mathrm{pr}_{K(1)}\, p = x^1$ and $\frac{\mathrm{pr}_{K(t)}\, p}{\| \mathrm{pr}_{K(t)}\, p \|} = x^t$ for all $t = 2, 3, \ldots$. We take the price and non-negative wealth transfer domain as $\boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}} = \{p \in \boldsymbol{R}^{\infty} | \exists [(x^t)_{t=1}^{\infty}] \in \Delta_{++}, \mathrm{pr}_{K(1)}\, p = x^1, \frac{\mathrm{pr}_{K(t)}\, p}{\| \mathrm{pr}_{K(t)}\, p \|} = x^t, \text{ for each } t = 1, 2, \ldots\} \times \{M | M : \boldsymbol{\mathcal{E}con}^* \ni \boldsymbol{\mathcal{E}} \mapsto M_{\boldsymbol{\mathcal{E}}} \in R_{+}^{\boldsymbol{N}}\}$. The excess demand function of the i-th consumer, $(\succsim_i, \omega_i)$, in $\boldsymbol{\mathcal{E}} \in \boldsymbol{\mathcal{E}con}^*$ is defined as $e_i : \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}} \ni (p, M) \mapsto e_i(p, M_{\boldsymbol{\mathcal{E}}}) \in \boldsymbol{R}_{\infty}$, where $e_i(p, M_{\boldsymbol{\mathcal{E}}})$ is the $\succsim_i$-greatest point in $\{x_i \in \boldsymbol{R}^{K_t} | p \cdot x_i \leqq p \cdot \omega_i + M_{\boldsymbol{\mathcal{E}}}(i)\}$, for each $i \in I_t$ and $t \in \boldsymbol{N}$.

Define $e : \boldsymbol{\mathcal{E}con}^* \times (\boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}}) \to \boldsymbol{R}^{\infty}$ by $e(\boldsymbol{\mathcal{E}}, \succsim_i, \omega_i, p, M) = (e_i(p, M_{\boldsymbol{\mathcal{E}}})_{i \in I_t})_{t \in \boldsymbol{N}}$. If for each $\boldsymbol{\mathcal{E}} \in \boldsymbol{\mathcal{E}con}^*$, $\pi(\boldsymbol{\mathcal{E}})$ denotes the set of all price-money equilibrium messages, then $(\boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}}, \pi, e)$ is a private representation of the WPO-IR compatible social choice correspondence $\boldsymbol{\mathcal{MW}alras}(\boldsymbol{\mathcal{E}})$. It is called the *price-money message mechanism.* Note that $(\boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}}, \pi, e)$ does not satisfy Axioms S and L.[8]

Theorem 2 (Price-Money Dictionary Theorem under Axiom L) : If (A, μ, f) is a private representation of the weak Pareto-optimal and individually rational social choice correspondence g, and if (A, μ, f) satisfies Axioms S and L, then (i) there exists a unique function $\phi : A \to \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}}$, such that the following triangle commutes, and (ii) on $\phi(A) \subset \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}}$, the monetary message mechanism satisfies Axioms S and L.

[7] It is possible to weaken Axiom L by restricting the condition to the messages satisfying $a \in \mu(\boldsymbol{\mathcal{E}})$. as long as we use the axiom with Axiom S. From the viewpoint of independency among axioms, it is desirable to define axioms L and M as conditions not on equilibria but merely on responses. Note that if Axiom L is satisfied, the sub-correspondence of a social choice correspondence g that is defined by the set of allocations under g of $\boldsymbol{\mathcal{E}}$ relating to a certain message $a \in \mu(\boldsymbol{\mathcal{E}})$ as its responses, also satisfies the LI condition.

[8] For Axiom L, see (2) of Theorem 1. For Axiom S, see footnote 7 of Urai and Murakami (2015b).



Proof: (i) Assume that (A, μ, f) is a private representation of g satisfying Axioms S and L, and let a be an element of A. Define for each $t \in \boldsymbol{N}$, $h^{(t)}(x, \succsim_i)$ for each consumption $x \in \boldsymbol{R}^{K_s}$ for agent $i \in I_s \subset I(t)$ of an economy $\mathcal{E} \in \mathcal{E}con^*$ as $h^{(t)}(x, \succsim_i) = \{p \in \Delta^{K(t)} | \, y \succ_i x \text{ implies } p \cdot y \geqq p \cdot x\}$, where every $\boldsymbol{R}^{K_s}$ is canonically identified with a subspace of $K(t)$. We first show that $\bigcap h^{(t)}(f_i(\mathcal{E}, a), \succsim_i)$ is non-empty for each $t \in \boldsymbol{N}$, where the intersection is over all consumers and economies in $\mathcal{E}con^*$, and $f_i(\mathcal{E}, a)$ is a response of $i \in I(t)$ in $\mathcal{E}$ to message a in (A, μ, f). Because $\Delta^{K(t)}$ is compact, and because each of the sets in the collection of which we are forming the intersection is closed, it is sufficient to show that $\bigcap_{s=1}^m h^{(t)}(f_{i_s}(\mathcal{E}_s, a), \succsim_{i_s})$ is non-empty for any $[(i_1, \mathcal{E}_1), (i_2, \mathcal{E}_2), \ldots, (i_m, \mathcal{E}_m)]$. Given the list $[(i_1, \mathcal{E}_1), (i_2, \mathcal{E}_2), \ldots, (i_m, \mathcal{E}_m)]$ of agents in $I(t)$ and economies, by Axiom S there exists $\mathcal{E}_* \in \mathcal{E}con^*$ containing $\{i_1, i_2, \ldots, i_m\}$ and $a \in \mu(\mathcal{E}_*)$, such that the equilibrium list, $(f_i(\mathcal{E}_*, a))_{i=1}^\infty$, is an extension of $(f_{i_s}(\mathcal{E}_s, a))_{s=1}^m$. Because $(f_i(\mathcal{E}_*, a))_{i=1}^\infty$ is an element of $g(\mathcal{E})$, the allocation is weakly Pareto-optimal, so by Balasko and Shell (1980) and Esteban (1986), it is supported by a price as a price-wealth equilibrium, and thus $\bigcap_{s=1}^m h^{(t)}(f_{i_s}(\mathcal{E}_s, a), \succsim_{i_s})$ is non-empty. Moreover, because for some economy and its agents, $\bigcap_{t=1}^\infty h^{(t)}(f_i(\mathcal{E}, a), \succsim_i)$ is singleton and is an element of $\Delta_{++}^{K(t)}$, it follows that $\bigcap h^{(t)}(f_i(\mathcal{E}, a), \succsim_i)$ is composed of a single point $p(t)$.

By definition of $h^{(t)}$, for all $t' \leqq t$, $p(t') = \varrho_{t't}(p(t))$, and we obtain an unique element $p \in \mathcal{P}$ by identifying it with the unique element of the projective limit $\varprojlim \bigcap_{s=1}^m h^{(t)}(f_{i_s}(\mathcal{E}_s, a), \succsim_{i_s}) \subset \Delta_{++}$. Let us denote that point, p, by $\phi^1(a)$, and define $\phi^2(a) = M$, $M : \mathcal{E}con^* \ni \mathcal{E} \mapsto M_{\mathcal{E}} \in \boldsymbol{R}^{\boldsymbol{N}}$, as $M_{\mathcal{E}}(i) = \phi^1(a) \cdot (f_i(\mathcal{E}, a) - \omega_i)$, which will be proved as non-negative in the following by Axioms S, L and Theorem 1. Let $\phi(a)$ be $(\phi^1(a), \phi^2(a)) \in \mathcal{P} \times \mathcal{M}$. In order to establish the theorem, it is sufficient to show that for each economy $\mathcal{E}_* \in \mathcal{E}con^*$ and $a \in A$, an allocation $y^* = (y_i^*)_{i=1}^\infty = (f_i(\mathcal{E}_*, a))_{i=1}^\infty$ is such that for each i, $y_i^* = f_i(\mathcal{E}_*, a)$ satisfies $M_{\mathcal{E}_*}(i) = \phi^1(a) \cdot (f_i(\mathcal{E}_*, a) - \omega_i) \geqq 0$. Fix a member i of $\mathcal{E}_*$. By using Axiom S, let $\mathcal{E}_{**}$ be an economy including i such that a is an equilibrium message for $\mathcal{E}_{**}$ and the response $y_i^{**} = f_i(\mathcal{E}_{**}, a)$ is equal to $y_i^* = f_i(\mathcal{E}_*, a)$. Without loss of generality we can assume that $\mathcal{E}_{**}$ has, at least to the generation s of member i, one consumer in each generation whose supporting hyperplane for any better set at their individually rational point is unique.[9] We show that response $y^{**} = f(\mathcal{E}_{**}, a)$ is a monetary Walras allocation. This allocation is IR and WPO. Moreover the social choice rule defined by a and its responses for all economies in $\mathcal{E}con^*$ having the same indices of agents in $\mathcal{E}$ satisfies the LI condition by Axiom L (see footnote 7). So by (i) of Theorem 1, the allocation belongs to $\mathcal{MW}alras(\mathcal{E}_{**})$.

(ii) One can observe in the above argument, $y^{**} = f(\mathcal{E}_{**}, a) = e(\mathcal{E}_{**}, \phi(a))$ is a monetary Walras allocation, which proves that Axiom S is satisfied on $\phi(A)$. Moreover, it is straightforward that the commutativity of the diagram with Axiom L for (A, μ, f) means that Axiom L is satisfied on $\phi(A)$. ■

By Theorem 2, from every private representation (A, μ, f) of social choice correspondence g satisfying Axiom S and L, there exists a unique *price dictionary function*, $\phi : A \to \mathcal{P} \times \mathcal{M}$. In other words, the

[9] It is always possible to add finite agents in constructing economy $\mathcal{E}_*$ in Axiom S.

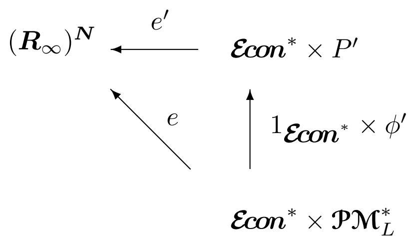
result of such *universal* private representations can be realized *efficiently* through the price-money message mechanism $(\mathcal{P} \times \mathcal{M}, \pi, e)$. Thus we have obtained the price-money dictionary theorem as in our previous paper (Theorem 2 of Urai and Murakami 2015b).

We can also obtain an isomorphism theorem for the price-money message mechanism (Theorem 3 of Urai and Murakami 2015b) as follows. Denote by $\mathcal{PM}_L^*$ the set of all $(p, M) \in \mathcal{P} \times \mathcal{M}$ which is an image of ϕ for some (A, μ, f) in Theorem 2 satisfying Axioms S and L. The following axiom on the dependence of monetary messages on the economic structure is necessary to show the second assertion.

Axiom D (Dependency on the Economic Structure): If $\mathcal{E} = (\{I_t\}_{t=1}^\infty,\ \{K_t\}_{t=1}^\infty,\ \{(\succsim_i,\ \omega_i)_{i \in \bigcup_{t \in \boldsymbol{N}} I_t}\})$ and $\mathcal{E}' = (\{I'_t\}_{t=1}^\infty,\ \{K'_t\}_{t=1}^\infty,\ \{(\succsim'_i,\ \omega'_i)_{i \in \bigcup_{t \in \boldsymbol{N}} I_t}\})$ are such that $\{I_t\}_{t=1}^\infty = \{I'_t\}_{t=1}^\infty$, $\{K_t\}_{t=1}^\infty = \{K'_t\}_{t=1}^\infty$ and $\omega_i = \omega'_i$ for all $i \in \bigcup_{t \in \boldsymbol{N}} I_t$, then $M_{\mathcal{E}} = M_{\mathcal{E}'}$ for all $M \in \mathcal{M}$.

Theorem 3 (Isomorphism Theorem under Axiom L): Consider the restriction of price-money message mechanism $(\mathcal{PM}_L^*, \pi, e)$. Let (P', π', e') be a private representation of the WPO-IR compatible social choice correspondence g on $\mathcal{E}con^*$. If (P', π', e') satisfies Axioms S and L, and if, for every private representation (A, μ, f) satisfying Axioms S and L, there exists a unique mapping $\phi' : A \to P'$ such that $f(\mathcal{E}, a) = e' \circ [1_{\mathcal{E}con^*} \times \phi'](\mathcal{E}, a)$, then (i) there exists an isomorphism (bijection) h' such that $h' : \mathcal{PM}_L^* \to P'$ and $e = e' \circ [1_{\mathcal{E}con^*} \times h']$. Moreover, (ii) assume that monetary messages satisfy Axiom D. If we can restrict the problem on spaces with topological (resp. on each inverse-system component space with differentiable) structures and continuous mappings (resp. differentiable coordinate mappings), then the isomorphism can be taken as the homeomorphism (resp. diffeomorphism for each component space).[10]

Proof: Because $(\mathcal{PM}_L^*, \pi, e)$ is now assumed to be a private representation of the WPO-IR compatible social choice correspondence g satisfying Axioms S and L, we have the next diagram by assumption.



Moreover, because (P', π', e') is also a private representation of the social choice correspondence g, the previous theorem shows that we have the next diagram.

[10] In this paper, the price-money message space has been treated as an inverse limit of finite dimensional domains of coordinate functions of e. The differentiability for e and a differentiable structure on its domain, however, is appropriate to be treated on each of its coordinate function, e_i, whose domain is always possible to be identified with a finite dimensional subspace of $\mathcal{P} \times \mathcal{M} \supset \mathcal{PM}_L^*$. More precisely, under the definitions of $e = (e_i)_{i=1}^\infty$ and $e' = (e'_i)_{i=1}^\infty$ with Axiom D, the bijection h' gives an algebraic isomorphism between the domain of e_i and e'_i for each i, to which the diffeomorphism argument can be applied. We can construct (as a subspace of $\mathcal{PM}_L^*$ under the identification of $\boldsymbol{R}_\infty \subset \boldsymbol{R}^\infty$) a direct limit of the finite dimensional projection, P_t, of the domain of $(e_i)_{i \in I(t)}$ for each $t \in \boldsymbol{N}$, so that the bijection h' gives an algebraic isomorphism between the domains P_t of $(e_i)_{i \in I(t)}$ and P'_t of $(e'_i)_{i \in I(t)}$ for each t. In this sense, each restriction of h' gives a diffeomorphism between the direct systems, $(P_t)_{t=1}^\infty$ and $(P'_t)_{t=1}^\infty$.

$$
\begin{array}{ccc}
(\boldsymbol{R}_\infty)^{\boldsymbol{N}} & \xleftarrow{\;e\;} & \mathcal{E}con^* \times \mathcal{PM}_L^* \\
 & \nwarrow^{e'} & \uparrow 1_{\mathcal{E}con^*} \times \phi \\
 & & \mathcal{E}con^* \times P'
\end{array}
$$

Since the identity mapping is the unique mapping for P' to P' satisfying $e' = e' \circ id$ and $\mathcal{PM}_L^*$ to $\mathcal{PM}_L^*$ satisfying $e = e \circ id$, we have $\phi' \circ \phi = id$ and $\phi \circ \phi' = id$, which means that ϕ and ϕ' are bijectives. Let us define h' as $h' = \phi'$, then we have the first assertion.

For the second assertion, for each $(p, M) \in \mathcal{PM}_L^*$, for each economy $\mathcal{E} \in \mathcal{E}con^*$ and for each generation t, consider two agents i_s and j_s, $s = 1, \ldots, t$ such that $e = (\cdots, e_{i_1}, \cdots, e_{j_1}, \cdots, e_{i_2}, \cdots, e_{j_2}, \cdots)$ on $\mathcal{P} \times \mathcal{M}$ is one to one, continuous and/or differentiable.[11] Then, the continuity (resp. differentiability) at (p, M) of h' will be ensured by the continuity (resp. differentiability) of e and e'. ■

Hence, Theorem 3 asserts that if we restrict the domain of the price-money messages to where Axioms S and L is satisfied, the the price-wealth formed monetary message mechanism is essentially the only object having the above universality and efficiency as a solution to the universal mapping problem.

The above price-money dictionary theorem (Theorem 2) and the isomorphism theorem (Theorem 3) can also be obtained through the monotonicity axiom (Axiom M) instead of the local independency axiom (Axiom L).

Theorem 4 (Price-Money Dictionary Theorem under Axiom M) : If (A, μ, f) is a private representation of the weak Pareto-optimal and individually rational social choice correspondence g, and if (A, μ, f) satisfies Axioms S and M, then (i) there exists a unique function $\phi : A \to \mathcal{P} \times \mathcal{M}$, such that the following triangle commutes, and (ii) on $\phi(A) \subset \mathcal{P} \times \mathcal{M}$, the monetary message mechanism satisfies Axioms S and M.

$$
\begin{array}{ccc}
\boldsymbol{R}^\infty & \xleftarrow{\;e\;} & \mathcal{E}con^* \times (\mathcal{P} \times \mathcal{M}) \\
 & \nwarrow^{f} & \uparrow 1_{\mathcal{E}con^*} \times \phi \\
 & & \mathcal{E}con^* \times A
\end{array}
$$

Proof: (i) We can repeat the argument in the first paragraph in the proof of Theorem 2 and obtain the single point $p(t)$ in $\bigcap_{1=1}^{\infty} h^{(t)}(f_i(\mathcal{E}, a), \succsim_i) \subset \Delta_{++}^{K(t)}$. Also by definition of $h^{(t)}$, for all $t' \leqq t$, we have $p(t') = \varrho_{t't}(p(t))$, and obtain an unique element $p \in \mathcal{P}$ by identifying it with the unique element of the projective limit $\varprojlim \bigcap_{s=1}^{m} h^{(t)}(f_{i_s}(\mathcal{E}_s, a), \succsim_{i_s}) \subset \Delta_{++}$. Let us define $\phi^1(a)$ as this unique element, p, and

[11] If generation $s \in \{1, \ldots, t\}$ of $\mathcal{E}$ consists of a single member, alternatively consider an economy $\hat{\mathcal{E}}$ including all members of genrations $1, \cdots, t$ of economy $\mathcal{E}$ such that every generation $s \in \{1, \ldots, t\}$ consists of at least two members and (p, M) is the equilibrium message of $\hat{\mathcal{E}}$ by Axiom S. To obtain a concrete example for such a one to one, continuous and/or differentiable mapping, take a pair of Cobb-Douglas and Leontief utility agents for each generation. Note that the Leontief type utility is not differentiable, but the demand function induced from it can be differentiable.

define $\phi^2(a) = M$, $M : \mathcal{E}con^* \ni \mathcal{E} \mapsto M_{\mathcal{E}} \in \boldsymbol{R}^{\boldsymbol{N}}$, as $M_{\mathcal{E}}(i) = \phi^1(a) \cdot (f_i(\mathcal{E}, a) - \omega_i)$, which will be proved as non-negative in the following by Axioms S and M.

Let $\phi(a)$ be $(\phi^1(a), \phi^2(a)) \in \mathcal{P} \times \mathcal{M}$. To establish the theorem, it is sufficient to show that for each economy $\mathcal{E}_* \in \mathcal{E}con^*$ and $a \in A$, an allocation $y^* = (y_i^*)_{i=1}^\infty = (f_i(\mathcal{E}_*, a))_{i=1}^\infty$ is such that for each i, $y_i^* = f_i(\mathcal{E}_*, a)$ satisfies $M_{\mathcal{E}_*}(i) = \phi^1(a) \cdot (f_i(\mathcal{E}_*, a) - \omega_i) \geqq 0$.

Assume the contrary, that is, there is a member i of $\mathcal{E}_*$ such that at $y_i^* = f_i(\mathcal{E}_*, a)$ we have $\phi^1(a) \cdot (f_i(\mathcal{E}_*, a) - \omega_i) < 0$. By using Axiom S, let $\mathcal{E}_{**}$ be an economy including i such that a is an equilibrium message for $\mathcal{E}_{**}$ and the response $y_i^{**} = f_i(\mathcal{E}_{**}, a)$ is equal to $y_i^* = f_i(\mathcal{E}_*, a)$, and there is at least one agent $j \neq i$ such that i and j are in the same generation s and the supporting hyperplane at $f_j(\mathcal{E}_{**}, a)$ for the better set of j is unique (that is necessarily equal to $\phi^1(a)$).

Fix the indifference surface of i in $\mathcal{E}_{**}$ at $y_i^* = y_i^{**} = f_i(\mathcal{E}_{**}, a)$ and change the preference of i to what is obtained through the homothetical transformation of the surface at y_i^*, $\precsim_i'$.[12]

Consider the economy $\mathcal{E}_{**}'$ such that the preference of i in $\mathcal{E}_{**}$ is replaced with $\precsim_i'$. By Axiom M, $f_i(\mathcal{E}_{**}', a) = y_i^*$. Change the preference of i to $\hat{\precsim}_i'$ so that $\hat{\precsim}_i'$ satisfies $\omega_i \hat{\precsim}_i' y_i^*$ and $y_i^* \not\hat{\precsim}_i' \omega_i$ and all assumptions in (OG3), and $\phi^1(a)$ remains to be a supporting hyperplane of the better set of i under $\hat{\precsim}_i'$ at y_i^*. Moreover, consider the homothetical transformation of the preference of i by using the indifference surface at y_i^* of $\hat{\precsim}_i'$, and call it $\precsim_i''$. Let us denote by $\mathcal{E}_{**}''$ the economy where we replace the preference $\precsim_i'$ of i with $\precsim_i''$.

By Axiom S, we have an economy $\mathcal{E}_{***}$ including i in $\mathcal{E}_{**}''$ and j in $\mathcal{E}_{**}$, and a is an equilibrium message for $\mathcal{E}_{***}$. Since $\phi^1(a)$ must support the better set of j at $f_j(\mathcal{E}_{**}, a)$ by WPO, $f_i(\mathcal{E}_{**}'', a)$ must be different from y_i^* by the condition IR, but must be equal to some point z_i at which the indifference surface of the point is supported by $\phi^1(a)$. That is, $f_i(\mathcal{E}_{**}'', a) = z_i \neq y_i^* = f_i(\mathcal{E}_{**}', a)$. This is a contradiction since $f_i(\mathcal{E}_{**}', a)$ is equal to $f_i(\mathcal{E}_{**}'', a)$ under Axiom M.

(ii) Repeat the arguments in the proof (ii) of Theorem 2 (replace Axiom L with Axiom M). ■

Denote by $\mathcal{PM}_M^*$ the set of all $(p, M) \in \mathcal{P} \times \mathcal{M}$ which is an image of ϕ for some (A, μ, f) in Theorem 4 satisfying Axioms S and M.

Theorem 5 (Isomorphism Theorem under Axiom M): Consider the restriction of price-money message mechanism $(\mathcal{PM}_M^*, \pi, e)$. Let (P', π', e') be a private representation of the WPO-IR compatible social choice correspondence g on $\mathcal{E}con^*$. If (P', π', e') satisfies Axioms S and M, and if, for every private representation (A, μ, f) satisfying Axioms S and M, there exists a unique mapping $\phi' : A \to P'$ such that $f(\mathcal{E}, a) = e' \circ [1_{\mathcal{E}con^*} \times \phi'](\mathcal{E}, a)$, then (i) there exists an isomorphism (bijection) h' such that $h' : \mathcal{PM}_M^* \to P'$ and $e = e' \circ [1_{\mathcal{E}con^*} \times h']$. Moreover, (ii) assume that monetary messages satisfy Axiom D and the set of all economies, $\mathcal{E}con^*$, consists of those having commodity structure $\{K_t\}_{t=1}^\infty$ and including at least two agents for each generation. If we can restrict the problem on spaces with topological (resp. on each inverse-system component space with differentiable) structures and continuous mappings (resp. differentiable coordinate mappings), then the isomorphism can be taken as the homeomorphism (resp. diffeomorphism for each component space).

Proof: Repeat the proof of Theorem 3 (replace Axiom L and $\mathcal{PM}_L^*$ respectively with Axiom M and

[12] Let $U(y_i^*)$ be the indifferent surface at y_i^* of i. Under the IR, $y_i^{**} \neq 0$. By the strict monotonicity, $\alpha U(y_i^*)$, $\alpha > 0$ covers the $\boldsymbol{R}_+^{K_s}$. Moreover for each $x \in \boldsymbol{R}_+^{K_s}$, there exist $\alpha(x)$ such that $x \in \alpha(x)U(y_*^i)$ and such $\alpha(x)$ is unique under the strong monotonicity. Thus by defining $u(x)$ as $u(x) = \alpha(x)$, we have the preference, $\precsim_i'$, satisfying all the conditions in (OG3).

$\mathfrak{PM}^*_M$). ■

We have thus obtained two kinds of the *price-money dictionary* theorems (Theorem 2 and 4) and the *isomorphism* theorems (Theorem 3 and 5). It can be said that the *price-money dictionary theorem*, which asserts that the price-money mechanism can be referenced *uniquely* and *universally* among the category satisfying the axioms in question (a property *for itself*), together with a restriction of price-money messages to the place where all of such axioms are satisfied (a property *on itself*), enable us to show the *isomorphism theorem* which says that the price-money message mechanism can be characterized as the *essentially* unique mechanism (up to isomorphism between the message spaces) in the category of mechanisms satisfying those axioms.

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