A Characterization of Potentials on a Network

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Abstract

In the network formation literature, Chakrabarti and Gilles (2007, Review of Economic Design, 11, 13-52) introduced a concept of network potentials. In this paper, we prove that a network payoff function admits a network potential function if and only if its payoff function coincides with the Shapley value of a particular class of cooperative games. We also examine when the set of potential maximizers coincides with the set of stable networks by the discrete optimization method. Similar to the game theory literature, for applications, we also show that potential methods is useful to select a particular stable network in terms of stochastic evolutionary process and robustness to incomplete information by Kajii and Morris (1997, Econometrica, 1283-1309).

JEL classification: C71, C72, C73, D85

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1 Introduction

The research of network formation keeps much attention in the economics recently. Each agent is involved in some network structures and then interact with them. The form of network plays an important role in the determination of many social outcomes. To investigate what kind of network configurations occur by the agent's interaction, Jackson

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and Wolinsky (1996) introduced an equilibrium concept so called pairwise stability. A network is pairwise stable if there is no agent who wants to sever the current link and there are no pair who want to make a new link. Jackson and Watts (2002) and Tercieux and Vannetelbosch (2006) considered one step further that which networks are stable in the long run by applying stochastic evolutionary dynamics. They showed that pairwise stable network is also stable in this sense.

Although it seems to have predictive power for the analysis, a pairwise stable network does not always exist. Also, there are possibilities that multiple pairwise stable networks exist. One of the class where a pairwise stable network exists is that agents' payoff functions admit a *potential* function. A potential function is a real valued function such that each agent's payoff difference between two networks coincides with that of the function. A potential function can be seen as the imaginary representative agent's payoff function in which each agent's payoff relevant information is aggregated. One of the usage of a potential function is that, once we find it, we can calculate which networks are stable by considering a simple maximization problem of that function. In the network formation literature, Chakrabarti and Gilles (2007) introduced this concept.

Potentials are widely used in physics and mathematics. In the non-cooperative game theory, Monderer and Shapley (1996) introduced this concept and established a class of potential games. Also, in the cooperative game theory, Hart and Mas-colell (1989) discovered the concept of potentials. Ui (2000) showed that these potentials can be unified and related with each other. The potential method is now widely used in both theoretical and applied field in the game theory.

The aim of this paper is to provide a new characterization for the class of network potentials. Our main theorem shows that there exists a network potential function if and only if agents' payoff functions are represented by a Shapley value of a particular class of cooperative games. Moreover, a network potential coincides with a potential of cooperative games. This result is a parallel to Ui (2000). Therefore, our result makes a new bridge between different fields. By using this result, we can give simple proof of a equivalence result of Chakrabarti and Gilles (2007), which showed that a network potential function exists if and only if the dual strategic game, which is called Myerson's consent game, is a potential game.

To examine the usage of network potential, we also demonstrate several results analogous to the game theory literature . First, by using the discrete concavity method developed by Ui (2008), a particular class of potential maximizers coincide with the set of pairwise stable networks. Hence, once we can know underling model fall into such a class, all of the stable networks can be found by the simple optimization problem. Second, embedding a model into stochastically dynamic process, we show that the potential maximizers has long run stability in the process. This result is similar to that of Jackson and Watts (2002) and Tercieux and Vannetelbosch (2006) [1]. Finally, we introduce a concept of robust stable network to information perturbation, which is based on Kajii and Morris (1997) and Ui (2002). By using the equivalence result between existence of a network potential and corresponding consent game being potential games, we show that the unique network potential maximizer is robust to information perturbation. This result can be seen as a strategic foundation of a cooperative concept of pairwise stability.

The rest of paper is organized as follows. In section 2, we give the model and notations. In section 3, we show our main results. In section 4, we provide a few examples. After that, we discuss when the set of maximizers of potentials coincide with the set of pairwise stable networks in section 5. In section 6, we discuss stochastic evolutionary result. In section 7, we introduce a robust stable network to information perturbation and show that the (unique) network potential maximizer is robust. Finally, we conclude in section 8.

2 Preliminaries

In this section, we introduce some concepts which are used in this paper.

2.1 Network

Let $N = \{1, \cdots, n\}$ be the (finite) set of players. Each distinct player $i \neq j$ is connected in some networks. The network relationship is described by a undirected graph whose nodes are players. A network g is the set of players who are linked with each other. Let $g^N = \{ij | i, j \in N, i \neq j\}$ be the set of all links. Then, a network g is considered as a subset of g^N, i.e., $g \subset g^N$. We denote $\mathbb{G}^N = \{g | g \subset g^N\}$ as the set of all the networks. For each network $g \in \mathbb{G}^N$ and player $i \in N$, let $N_i(g) = \{j \in N | i \neq j \text{ and } ij \in g\}$ be the set of i's neighborhood in g. For $S \in 2^N$ and $g \in \mathbb{G}^N, g|_S = \{ij \in g | i \in S \text{ and } j \in S\}$ is the restricted network on S. Also, g_S is a network constructed by agents in S. The set of them is denoted by $\mathbb{G}^S$. For $ij \in g, g - ij = g \backslash \{ij\}$ is the network which remains after removing an existing link ij. Similarly, for $ij \notin g, g + ij = g \cup \{ij\}$ is the network formed by adding the new link ij.

The utility of a network to player $i \in N$ is given by $\phi_i : \mathbb{G}^N \to \mathbb{R}$. Let we denote $\phi = (\phi_i)_{i \in N}$ as a vecter form, which we call network payoff function. Chakrabarti and Gilles (2007) introduced the special class of the network payoff function, which is called

[1] As we will discuss it later, our process and that of them is different from each other. Hence, our result is not directly implied by theirs.

network potential function. Intuitively, if a network payoff function ϕ admits a network potential function, then each player's payoff information is aggregated to the imaginary representative player's payoff function. The function is the analogy of the potential function defined by Monderer and Shapley (1996) in the non-cooperative game. The formal definition is as follows.

Definition 1. A network payoff function $\phi = (\phi_i)_{i \in N}$ on $\mathbb{G}^N$ admits a *network potential* if there is a function $\omega : \mathbb{G}^N \to \mathbb{R}$ such that $\forall g \in \mathbb{G}^N, \forall i \in N$ and $\forall ij \in L_i(g)$,

$$\phi_i(g) - \phi_i(g - ij) = \omega(g) - \omega(g - ij).$$

In the social network literature, we have used an equilibrium concept as a tool of analysis. The most typical one is called pairwise stability defined by Jackson and Wolinsky (1996). A network is pairwise stable if there is no one want to sever the link in which he is involved and there is no pair who agree to make a new link.

Definition 2. A network g is pairwise stable if

(i) for all $ij \in g, \phi_i(g) \geq \phi_i(g - ij)$ and $\phi_j(g) \geq \phi_j(g - ij)$, and

(ii) for all $ij \notin g$, if $\phi_i(g) < \phi_i(g + ij)$ then $\phi_j(g) > \phi_j(g + ij)$.

Note that it is easily verified that if a network payoff function ϕ admits a network potential function ω, then a maximizer of it corresponds to one of the pairwise stable network. If the number of players are finite, there are only finite number of networks. Hence, a maximizer of ω always exists, which implies that the existence of a pairwise stable network.

2.2 Shapley value

For each $S \in 2^N$, a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$ is called a characteristic function and (N, v) is called a cooperative game with transferable utility or a TU game. We denote $\mathscr{G}_N$ as the set of all TU games. For $v \in \mathscr{G}_N$ and $T \in 2^N$, let the restricted game $v|_T \in \mathscr{G}_N$ be such that

$$u|_T(S) = \begin{cases} v(S \cap T) & \text{if } \ S \cap T \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

For $T \in 2^N$, let us define a unanimity game $u_T \in \mathscr{G}_N$ so that

$$u_T(S) = \begin{cases} 1 & \text{if } \ T \subset S \\ 0 & \text{otherwise.} \end{cases}$$

Shapley (1953) shows a next decomposition result.

Lemma 1. *Any TU game $v \in \mathscr{G}_N$ is described by a unique linear combination of a collection of unanimity games $\{u_T\}_{T\in 2^N}$, i.e.,*

$$v(S) = \textstyle\sum_{T\in 2^N} v^T u_T(S)$$

where

$$v^T = \textstyle\sum_{T\subset R}(-1)^{|R\setminus T|}v(T).$$

The Shapley value is defined by the map $\psi : \mathscr{G}_N \to \mathbb{R}^N$ such that

$$\psi_i(v) = \textstyle\sum_{S\in 2^N, i\in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!}(v(S) - v(S\setminus\{i\})).$$

It is known that ψ is linear map which satisfies

$$\psi_i(u_T) = \begin{cases} 1/|T| & \text{if } \ i \in T \\ 0 & \text{otherwise.} \end{cases}$$

Then, we can write $\psi_i(v) = \sum_{T\in 2^N, i\in T} v^T/|T|$ where $v^T/|T|$ is called Harsanyi's dividend to the member of T.

In the TU game, Hart and Mas-colell (1989) defined a potential function. For a function $P : \mathscr{G}_N \to \mathbb{R}$, the marginal contribution of i to P, denoted as $DP_i : \mathscr{G}_N \to \mathbb{R}$, is defined as

$$DP_i(v) = P(v) - P(v|_{N\setminus\{i\}}).$$

Then, P is a *potential* if it satisfies

$$\textstyle\sum_{i\in N} DP_i(v) = v(N).$$

Hart and Mas-colell (1989) show that the potential P is uniquely given by $P(v) = \sum_{T\in 2^N} v^T/|T|$. Thus, each agent's marginal contribution satisfies

$$DP_i(v) = \textstyle\sum_{T\in 2^N, i\in T} v^T/|T| = \psi_i(v).$$

2.3 Potential games

For each $i \in N$, let A_i be the set of strategies and $u_i : A \to \mathbb{R}$ be the payoff function. We denote $\Gamma(N, A, u)$ as a strategic form game. Monderer and Shapley (1996) introduced a class of potential games. Formally, it is defined as follows.

Definition 3. A game $\Gamma(N, A, u)$ is called a *potential game* if there is a function $V : A \to \mathbb{R}$ such that $\forall i \in N, \forall a_i' \in A_i$ and $\forall a \in A$,

$$u_i(a_i', a_{-i}) - u_i(a) = V(a_i', a_{-i}) - V(a).$$

Let us consider a collection of TU games $\{v_a\}_{a \in A}$ such that $v_a(S) = v_a'(S)$ if $a_S = a_S'$, which is called a TU game with action choice. $\mathscr{G}_{N,A} = \{\{v_a\}_{a \in A} | v_a(S) = v_a'(S) \text{ if } a_S = a_S'\}$ denote the set of all TU games with action choices. Ui (2000) showed the next surprising relationship between potential games and the shapley value, hence the potential of a TU game.

Theorem 1. *Theorem 2 of Ui (2000)*
For any game $\Gamma(N, A, u)$, the following statements are equivalent:
(i) $\Gamma(N, A, u)$ is a potential game.
(ii) There exists $\{v_a\}_{a \in A}$ such that

$$u_i(a) = \psi_i(v_a) \text{ for all } i \in N.$$

A potential function V is given by

$$V(a) = P(v_a).$$

3 Main results

Our goal of this section is that we also show the relationship between a network potential and the Shapley value. To show it, let us consider a collection of TU games $\{v_g\}_{g \in \mathbb{G}^N}$ such that $v_g(S) = v_{g'}(S)$ if $g|_S = g'|_S$, which we call a TU game on network. Note that the value of each coalition $S \in 2^N$ is only defined by the network structure in it. We denote $\mathscr{G}_{N,\mathbb{G}^N} = \{\{v_g\}_{g \in \mathbb{G}^N} | v_g(S) = v_{g'}(S) \;\text{ if }\; g|_S = g'|_S\}$ as the set of all TU games on networks. By using the argument of Ui (2000), we show the next main representation theorem, which complements Theorem 3.7 in Chakrabarti and Gilles (2007).

Theorem 2. *For any network payoff function $\phi = (\phi_i)_{i \in N}$ on $\mathbb{G}^N$, the following statements are equivalent:*
(i) The network function ϕ admits a network potential.
(ii) There exists $\{v_g\}_{g \in \mathbb{G}^N}$ such that

$$\phi_i(g) = \psi_i(v_g) \text{ for all } i \in N.$$

A potential function ω is given by

$$\omega(g) = P(v_g).$$

This is a version of representation theorem of Ui (2000) in terms of a network potential. This result shows a new relationship between several different potential concept. To prove it, let us first consider the next lemmas. The first one is a network version of the decomposition lemma which is shown by Slade (1994) and Facchini et. al. (1997) in the potential game.

Lemma 2. *A network payoff function $\phi = (\phi_i)_{i\in N}$ on $\mathbb{G}^N$ admits a network potential if and only if there exist functions $\omega : \mathbb{G}^N \to \mathbb{R}$ and $\lambda_i : \mathbb{G}^{N\setminus\{i\}} \to \mathbb{R}$ such that $\forall g \in \mathbb{G}^N, \forall i \in N$,*

$$\phi_i(g) = \omega(g) + \lambda_i(g|_{N\setminus\{i\}})$$

where ω is a potential function.

Proof. ($\Leftarrow$) By the direct calculation, $\forall g \in \mathbb{G}^N, \forall i \in N$ and $\forall ij \in N_i(g)$, we obtain

$$\begin{aligned}\phi_i(g) - \phi_i(g - ij) =& (\omega(g) - \lambda_i(g|_{N\setminus\{i\}})) - (\omega(g - ij) - \lambda_i((g - ij)|_{N\setminus\{i\}})) \\ =& \omega(g) - \omega(g - ij)\end{aligned}$$

because $g|_{N\setminus\{i\}} = (g - ij)|_{N\setminus\{i\}}$.

($\Rightarrow$) Define $\lambda_i(g_{N\setminus\{i\}}) = \phi_i(g_{N\setminus\{i\}} + ij) - \omega(g_{N\setminus\{i\}} + ij)$ for any $g_{N\setminus\{i\}} \in \mathbb{G}^{N\setminus\{i\}}$ and $j \in N\setminus\{i\}$. By definition of the network potential, this value is well-defined, which completes the proof. □

Next lemma shows a property of Harsanyi's dividends of $\{v_g\}_{g\in\mathbb{G}^N}$.

Lemma 3. *$\{v_g\}_{g\in\mathbb{G}^N} \in \mathscr{G}_{N,\mathbb{G}^N}$ if and only if $g|_S = g'|_S$ implies $v_g^S = v_{g'}^S$ for any $S \in 2^N$.*

Proof. ($\Leftarrow$) Suppose $g|_S = g'|_S$ implies $v_g^S = v_{g'}^S$. Then, for any $S \in 2^N$,

$$v_g(S) = \textstyle\sum_{T\subset 2^N} v_g^T u_T(S) = \sum_{T\subset S} v_g^T.$$

Note that $g|_S = g'|_S$ implies $g|_T = g'|_T$ for any $T \subset S$. Thus, $v_g(S) = v_{g'}(S)$, which means that $\{v_g\}_{g\in\mathbb{G}^N} \in \mathscr{G}_{N,\mathbb{G}^N}$.

($\Rightarrow$) Suppose that $\{v_g\}_{g\in\mathbb{G}^N} \in \mathscr{G}_{N,\mathbb{G}^N}$. Note that

$$v_g^S = \textstyle\sum_{T\subset S} (-1)^{|S\setminus T|} v_g(T),$$

and, for all $T \subset S$, $v_g(T) = v_{g'}(T)$ if $g|_S = g'|_S$. Thus, $g|_S = g'|_S$ implies $v_g^S = v_{g'}^S$.

□

A collection $\{\zeta_S\}_{S\in 2^N}$ such that $\zeta_S : \mathbb{G}^S \to \mathbb{R}$ is called an interaction network potential, which is a analogous definition of interaction potential defined by Ui (2000). Then, we show the next theorem. Actually, this is the equivalent to the Theorem 1, which we prove later.

Theorem 3. *For any network payoff function $\phi = (\phi_i)_{i\in N}$ on $\mathbb{G}^N$, the following statements are equivalent:*

(i) The network function ϕ admits a network potential.

(ii) There exists an interaction network potential $\{\zeta_S\}_{S\in 2^N}$ such that

$$\phi_i(g) = \textstyle\sum_{S\in 2^N, i\in S} \zeta_S(g|_S) \text{ for all } i \in N.$$

A potential function ω is given by

$$\omega(g) = \textstyle\sum_{S\in 2^N} \zeta_S(g|_S).$$

Proof. $(i) \Leftarrow (ii)$. Let $\{\zeta_S\}_{S\in 2^N}$ be an interaction potential satisfying the conditions. Define $\omega(g) = \sum_{S\in 2^N} \zeta_S(g|_S)$. Then

$$\begin{aligned}
\omega(g) - \omega(g - ij) &= \sum_{S\in 2^N} \zeta_S(g|_S) - \sum_{S\in 2^N} \zeta_S((g - ij)|_S) \\
&= \sum_{S\in 2^N, i\in S} \zeta_S(g|_S) + \sum_{S\in 2^N, i\notin S} \zeta_S(g|_S) \\
&\quad - \sum_{S\in 2^N, i\in S} \zeta_S((g - ij)|_S) - \sum_{S\in 2^N, i\notin S} \zeta_S((g - ij)|_S) \\
&= \sum_{S\in 2^N, i\in S} \zeta_S(g|_S) - \sum_{S\in 2^N, i\in S} \zeta_S((g - ij)|_S) \\
&= \phi_i(g) - \phi_i(g - ij),
\end{aligned}$$

where the third equality follows from the observation that $g|_S = (g - ij)|_S$ if $i \notin S$. Thus, ϕ admits a network potential ω.

$(i) \Rightarrow (ii)$. Let ω be a network potential. By Lemma 2, let $\lambda_i(g|_{N\setminus\{i\}}) = \phi_i(g) - \omega(g)$. For $S \in 2^N$, let us define

$$\zeta_S(g_S) = \begin{cases} \omega(g) + \sum_{i\in N} \lambda_i(g|_{N\setminus\{i\}}) & \text{if } S = N \\ -\lambda_i(g|_{N\setminus\{i\}}) & \text{if } S = N\setminus\{i\} \text{ for some } i \\ 0 & \text{if } |S| \le |N| - 2. \end{cases}$$

Then each $i \in N, S \in 2^N$, and $g_S \in \mathbb{G}^S$,

$$
\begin{aligned}
\sum_{S \in 2^N, i \in S} \zeta_S(g_S) &= \sum_{j \in N \setminus \{i\}} \zeta_{N \setminus \{j\}}(g_{N \setminus \{j\}}) + \zeta_N(g) \\
&= - \sum_{j \in N \setminus \{i\}} \lambda_j(g|_{N \setminus \{j\}}) + \omega(g) + \sum_{j \in N} \lambda_j(g|_{N \setminus \{j\}}) \\
&= \omega(g) + \lambda_i(g|_{N \setminus \{i\}}) = \phi_i(g).
\end{aligned}
$$

Also, for each $S \in 2^N$ and $g_S \in \mathbb{G}^S$,

$$
\begin{aligned}
\sum_{S \in 2^N} \zeta_S(g_S) &= \sum_{j \in N} \zeta_N(g_{N \setminus \{j\}}) + \zeta_N(g) \\
&= - \sum_{j \in N} \lambda_j(g|_{N \setminus \{j\}}) + \omega(g) + \sum_{j \in N} \lambda_j(g|_{N \setminus \{j\}}) \\
&= \omega(g).
\end{aligned}
$$

Thus, this completes that proof. □

proof of Theorem 2. by Lemma 3, there is a one-to-one correspondence between $\{v_g\}_{g \in \mathbb{G}^N}$ and $\{\zeta_S\}_{S \in 2^N}$ such that

$$\zeta_S(g|_S) = v_g^S / |S|.$$

Then, by Theorem 3 and the same argument on the proof of Theorem 2 of Ui (2000), we establish the result. □

3.1 Relation between Myerson's consent game

Myerson (1991) introduced a model of strategic network formation game. This is called consent game. Given $\mathbb{G}^N$ and a profile of network payoff function $(\phi_i)_{i \in N}$, let $\mathcal{A}_i = \{(l_{ij})_{j \neq i} | l_{ij} \in \{0, 1\}\}$ be the set of action for agent i. For each element $l_i \in \mathcal{A}_i$, it means $l_{ij} = 1$ if i want to from a link with j. Let $g(l) = \{ij \in g^N | l_{ij} \cdot l_{ji} = 1\}$ be a induced network by action profile l. We call $\Gamma_\phi = (\mathcal{A}, \pi_\phi)$ consent game corresponding network payoff function ϕ such that $\pi_{\phi,i}(l) = \phi_i(g(l))$. Let $\mathcal{A}_g = \{l \in \mathcal{A} | g(l) = g\}$ be the set of strategy profile which induces the network g. Note that each strategy profile l induces the unique network $g(l)$, but there are many strategy profiles which induce the same network. We define $\hat{l}_g \in \mathcal{A}_g$ is the (unique) *non-superfluous* strategy profile, which is defined as for all pair i, j and $l_{g,ij} = 1$ if and only if $ij \in g$. In this subsection, by using our representation theorem, we give a simpler proof of Theorem 3.3 in Chakrabarti and Gilles (2007).

Theorem 4. *Theorem 3.3 of Chakrabarti and Gilles (2007)*
For any network payoff function $\phi = (\phi_i)_{i\in N}$ on $\mathbb{G}^N$, the following statements are equivalent:

(i) The network function ϕ admits a network potential.

(ii) The consent game Γ_ϕ admits a potential.

Proof. $(i) \Rightarrow (ii)$. By Theorem 2, let $\{v_g\}_{g\in\mathbb{G}^N}$ be a TU game on networks corresponding the network payoff function $\phi = (\phi_i)_{i\in N}$. For any $l \in \mathcal{A}$, let $v_l = v_{g(l)}$. Note that for any $l \in \mathcal{A}, l_S = l'_S$ implies $g(l)|_S = g(l')|_S$ because $g|_S$ is not affected by the actions of $N\backslash S$ for all $g \subset g^N$ and $S \in 2^N$. Then, this implies that $v_{g(l)}(S) = v_{g(l')}(S)$. Hence, $\{v_l\}_{l\in\mathcal{A}}$ constitutes a TU game with action choice and which satisfies $\pi_i(l) = \phi_i(g(l)) = \psi_i(v_{g(l)}) = \psi_i(v_l)$. By Theorem 1, we obtain the result.

$(i) \Leftarrow (ii)$. By Theorem 1, let $\{v_l\}_{l\in\mathcal{A}}$ be a TU game with action choice corresponding the potential game Γ_ϕ. Let us define the function $f : \mathbb{G}^N \to \mathcal{A}$ such that $f(g) = \hat{l}_g$ for all $g \in \mathbb{G}^N$. This function is well-defined because $\hat{l}_g$ is unique for all $g \in \mathbb{G}^N$. Next, define $v_g = v_{f(g)}$ for all $g \in \mathbb{G}^N$. By definition of $\hat{l}_g, g|_S = g'|_S$ implies $\hat{l}_g|_S = \hat{l}_{g'}|_S$. Then, this implies that $v_g(S) = v_{g'}(S)$ by construction. Hence, $\{v_g\}_{g\in\mathbb{G}^N}$ constitutes a TU game on networks and which satisfies $\phi_i(g) = \phi_i(g(\hat{l}_g)) = \pi_i(\hat{l}_g) = \psi_i(v_{\hat{l}_g}) = \psi_i(v_g)$. By Theorem 2, we obtain the result and it completes the proof. □

4 Examples

We give examples to show applicability of the potential function.

Example 1: Connection model

Jackson and Wolinsky (1996) gave the next example for the network formation. Let $\delta \in (0, 1)$ be a discount rate and c_{ij} be a link formation cost between i and j. We consider the following type of payoff function:

$$\phi_i^\delta(g) = \sum_{j\neq i} \delta^{t_{ij}} - \sum_{j\in N_i(g)} c_{ij}$$

where

$$t_{ij} = \begin{cases} 1 & \text{if } \ j \in N_i(g) \\ \infty & \text{otherwise.} \end{cases}$$

Suppose that cost is symmetric i.e., $c_{ij} = c_{ji}$ for all $i, j \in N$. Let $\omega(g) = \frac{1}{2}\sum_{i\in N}\phi_i^\delta(g)$. We claim that ω is a potential function. Indeed, by the calculation, for each $g \in \mathbb{G}^N$ and $ij \in g$,

$$\begin{aligned}\omega(g) - \omega(g - ij) =& \frac{1}{2}\sum_{k\in N}(\phi_k^\delta(g) - \phi_k^\delta(g - ij))\\ =& \frac{1}{2}(\phi_i^\delta(g) - \phi_i^\delta(g - ij) + \phi_j^\delta(g) - \phi_j^\delta(g - ij))\\ =& \frac{1}{2}(\delta + c_{ij} + \delta + c_{ji})\\ =& \delta + c_{ij}\\ =& \phi_i^\delta(g) - \phi_i^\delta(g - ij).\end{aligned}$$

Thus, ω is a potential function for ϕ.

Example 2: Neural network

Ui (2000) demonstrated that neural network model is mathematically similar structure of the potential game where players are neurons and strategies are firing or not firing. We give reinterpretation of the model in terms of network formation. Consider the following form of payoff function:

$$\phi_i(g) = \sum_{j\in N\setminus i} g_{ij} f(w_i, w_j) - c_i(w_i)$$

where

$$g_{ij} = \begin{cases} 1 & \text{if} \quad ij \in g \\ 0 & \text{otherwise,} \end{cases}$$

$f : \mathbb{R}^2 \to \mathbb{R}$ and $c_i : \mathbb{R} \to \mathbb{R}$. Each $w_i \in \mathbb{R}$ can be considered as the state or energy of neuron. If i and j are connected, then a signal $f(w_i, w_j)$ is transmitted between each other. $c_i(\cdot)$ capture a cost of energy. In this sense, this model can be considered as the formation of neural network. Let $\omega(g) = \frac{1}{2}\sum_{i\in N}\phi_i(g)$. By the similar calculation in Example 1, this is a network potential function for ϕ.

Example 3: Social distance[2]

Iijima and Kamada (2014) consider how agents' own characteristic affects the resulting network configurations. In their model, each agent i has his own multidimensional (m-dimension) characteristic, which is called type, $x_i = (x_{i1}, \cdots, x_{im}) \in [0, 1]^m$ and their is a

[2]We thank Ryota Iijima for suggesting this model.

measure how agent i and j is related/similar in terms of their type, which is called *social distance*[3] $d : X \times X \to \mathbb{R}$. They assume that each agent's payoff from the network is as follows:

$$\phi_i(g) = \left(\sum_{j \in N_i(g)} b(d(x_i, x_j)) \right) - c(q_i)$$

where $b(\cdot) > 0$ is a weakly decreasing, left-continuous function, $c(\cdot)$ is a strictly increasing function and $q_i = |N_i(g)|$. Note that $b(\cdot), c(\cdot)$ are symmetric among all agents. Thus, $\omega(g) = \frac{1}{2} \sum_{i \in N} \phi_i(g)$ is a potential function in this case.

5 Potential maximizers and pairwise stable networks

In this section, we consider when the set of maximizers of a network potential function is equivalent to the set of pairwise stable networks. In Ui (2008), he shows the set of maximizers of a potential function in the game where action sets are discrete coincide with the set of Nash equilibria by the discrete concavity property. We show this method can be applicable for a network potential function.

Following Ui (2008), we introduce some more notations. For a finite set M, let $X \subset \mathbb{Z}^M$ be a discrete space where $X = \Pi_{i \in M} X_i, X_i = \{x_i \in \mathbb{Z} | \underline{x_i} \leq x_i \leq \overline{x_i}\} \subset \mathbb{Z}$ and $\underline{x_i}, \overline{x_i} \in \mathbb{Z} \cup \{-\infty, +\infty\}$. Let $||x|| = \sum_{i \in M} |x_i|$ be the l_1-norm of a vector $x \in \mathbb{Z}^M$. We say that a function $f : X \to \mathbb{R}$ satisfies the *larger midpoint property (LMP)* if , for any $x, y \in X$ with $||x - y|| = 2$,

$$\max_{z \in X : ||x-z|| = ||y-z|| = 1} f(z) \geq t f(x) + (1-t) f(y) \quad (\exists t \in (0,1))$$

In the definition, the midpoint of each $x, y \in X$ with $||x - y|| = 2$ is the points $z \in X$ satisfying $||x - z|| = ||y - z|| = 1$. Ui (2008) shows the following result, which states when both local and global optimality coincide.

Proposition 1. *Proposition 1 of Ui (2008)*
Suppose that $f : X \to \mathbb{R}$ satisfies LMP. Then, $f(x) \geq f(y)$ for all $y \in X$ with $||x - y|| \leq 1$ if and only if $f(x) \geq f(y)$ for all $y \in X$.

To use this result, let $M = \frac{N(N-1)}{2}, X_i = \{0, 1\}$ and we identify $\mathbb{G}^N$ as $X = \{0, 1\}^{\frac{N(N-1)}{2}}$. So, each network can be regarded as the finite dimensional vector in which each element is a potential link. In the network setting, for each $g, g' \in \mathbb{G}^N$ with $||g - g'|| = 2$, it is easily seen that there are always two midpoint $\tilde{g}_1, \tilde{g}_2$. We state the following proposition.

[3] d is metric in usual sense.

Proposition 2. *Let ϕ admits a network potential ω. Suppose that ω satisfies LMP. Then, $g \in \mathbb{G}^N$ maximizes ω if and only if it is a pairwise stable network.*

Proof. Since only if part is already said in the section 2, it suffices to show if part. Let g be a pairwise stable network. By definition, we must have

$$\omega(g) - \omega(g - ij) \geq 0 \text{ for } ij \in g \text{ and } \omega(g) - \omega(g + ij) \geq 0 \text{ for } ij \notin g.$$

Since $||g - (g - ij)|| = ||g - (g + ij)|| = 1$ for all $ij \in g$ $(ij \notin g)$, by Proposition 1, $\omega(g) \geq \omega(g')$ for all $g' \in \mathbb{G}^N$. This means that g, a pairwise stable network, maximizes ω. $\square$

Thus, once we know that the underling situation can be described by the payoff function ϕ which admits a potential function satisfying LMP, we can know all the pairwise stable networks by computing the maximizers of a potential function. The number of networks are always finite and so calculation of pairwise stable networks can be implemented computationally.

6 Stochastic stability

In this section we show one of a usage of network potentials in terms of stochastically evolutionary process. Jackson and Watts (2002) and Tercieux and Vannetelbosch (2006) demonstrated the stability of a network in the stochastic evolutionary process initiated to Kandori et.al. (1993) and Young (1993). They use uniform mistake and bilateral link formation process to show the long run stability of a pairwise stable network. We use a slight different assumption of mistake, which is called logit dynamics and unilateral link formation process. The logit dynamics is studied by Blume (1993) and recently by Als-Ferrer and Netzer (2010) and Sawa (2014). Although our method is technically different of previous study of Jackson and Watts (2002) and Tercieux and Vannetelbosch (2006), we show the similar result in terms of network potentials.

We consider the infinite horizon discrete time process $t \in \{1, 2, \cdots\}$. In each period, one of the player i is drawn from the population with probability $p(i) \in (0.1)$. We assume that $p(i) = \frac{1}{N}$ for each $i \in N$, i.e. uniform distribution. For each $g \in \mathbb{G}^N$, let $Ad^i(g) = \{g' \in \mathbb{G}^N | g' = g + ij \text{ or } g' = g - ij \text{ for some } j \neq i\ \}$ be the set of adjacent networks of g for i. Define $Ad(g) = \cup_{i \in N} Ad^i(g)$. Also, for each $g \in \mathbb{G}^N$ and $g' \in Ad(g)$, define (potentially) movers g to g', $R(g, g') : \mathbb{G}^N \times \mathbb{G}^N \rightrightarrows 2^N$ such that $R(g, g') = \{i, j\}$ where $g' = g + ij$ or $g' = g - ij$.

Unperturbed dynamics

In each period, one agent is drawn from the population. He can choose a new partner or resolve a relationship with one of the current partner. He can choose a better one than current situation. So, given the current network g, the transition probability to $g' \in Ad(g)$ is as follows: $P^0_{g,g'} = \sum_{k \in R(g,g')} p(k) q^0_k(g,g')$ where, for all $i \in N$,

$$q^0_i(g,g') = \begin{cases} 1 & \text{if} \quad \phi_i(g') - \phi_i(g) > 0, \text{ and } g' \in Ad^i(g) \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

If there are indifferent choice, he chooses them equally likely. This process is essentially same as the *better response dynamics* in Cabrales and Serrano (2011,2012) where the feasible action for each agent is either to form a new link or to sever a current link and the a Markov chain for which state space is $\mathbb{G}^N$. For the transition, given payoff structure, only the player selecting probability and initial condition determines the asymptotic distribution. Thus, this is a deterministic process.

Stochastic perturbation under logit-response dynamics

We consider the logit-response perturbation from the deterministic process. Let $q^\epsilon_i(g,g') = \frac{exp[\epsilon^{-1}\phi_i(g')]}{\sum_{g'' \in \{g',g\}} exp[\epsilon^{-1}\phi_i(g'')]}$ for each $\epsilon \in (0,1)$ and for all $i \in N$ and $P^\epsilon_{g,g'} = \sum_{k \in R(g,g')} p(k) q^\epsilon_k(g,g')$. This is the perturbed Markov chain for the underlying process. Since this process is irreducible and aperiodic, it has the unique stationary distribution μ^ϵ. As ϵ goes to zero, this stationary distribution converges to corresponding long run distribution μ. A network g is *stochastically stable* if it is in the support of the limiting (as $\epsilon \to 0$) distribution i.e., $g \in supp(\mu)$.

Stochastic stability

If ϕ admit a network potential, then by Lemma 2, we can write $\phi_i(g) = \omega(g) + \lambda_i(g|_{N\setminus\{i\}})$ for each $i \in N, g \in \mathbb{G}^N$. Note that if $g' \in Ad^i(g) \cap Ad^j(g)$ then $g|_{N\setminus\{i\}} = g'|_{N\setminus\{i\}}$ and $g|_{N\setminus\{j\}} = g'|_{N\setminus\{j\}}$. Therefore, if ϕ admits a network potential, we can write $q^\epsilon_i(g,g') = \frac{exp[\epsilon^{-1}\omega(g')]}{\sum_{g'' \in \{g',g\}} exp[\epsilon^{-1}\omega(g'')]}$. Hence, $P^\epsilon_{g,g'} = \frac{2}{N} \frac{exp[\epsilon^{-1}\omega(g')]}{\sum_{g'' \in \{g',g\}} exp[\epsilon^{-1}\omega(g'')]}$. Following proposition gives a simple form of stationary distribution in terms of a potential function.

Proposition 3. *Suppose that ϕ admits a network potential ω. Then, for each $\epsilon > 0$ the unique stationary distribution of logit-response dynamics μ^ϵ is given by*

$$\mu^\epsilon(g) = \frac{exp[\epsilon^{-1}\omega(g)]}{\sum_{g' \in \mathbb{G}^N} exp[\epsilon^{-1}\omega(g')]}.$$

Proof. It is enough to show that the distribution μ^ϵ satisfies the *detailed balance condition*, i.e. $\mu^\epsilon(g)P^\epsilon_{g,g'} = \mu^\epsilon(g')P^\epsilon_{g',g}$ for all $g, g' \in \mathbb{G}^N$. Note that if two networks are not adjacent, then the transition probability is zero and

$$\frac{P^\epsilon_{g,g'}}{P^\epsilon_{g',g}} = \frac{exp[\epsilon^{-1}\omega(g')]}{exp[\epsilon^{-1}\omega(g)]} = \frac{exp[\epsilon^{-1}\omega(g')]}{\sum_{g''\in\mathbb{G}^N} exp[\epsilon^{-1}\omega(g'')]} \times \frac{\sum_{g''\in\mathbb{G}^N} exp[\epsilon^{-1}\omega(g'')]}{exp[\epsilon^{-1}\omega(g)]} = \frac{\mu^\epsilon(g')}{\mu^\epsilon(g)},$$

so, $\mu^\epsilon(g)P^\epsilon_{g,g'} = \mu^\epsilon(g')P^\epsilon_{g',g}$ for all $g, g' \in \mathbb{G}^N$. □

By this result, we give the characterization for the stochastically stable network in terms of a network potential.

Theorem 5. *Suppose that ϕ admits a network potential ω. Then, a network g is stochastically stable if and only if g maximizes ω.*

Remark. As we have already known, the network potential maximizer is (strictly) pairwise stable[4]. Note that our dynamics do not assume that bilateral network formation, which is the requirement for the pairwise stability. However, our result show that even unilateral link formation process can be reached to the stable network under the logit-dynamics with a potential function. This result gives a new insight to the line of study the stochastic evolutionary selection of the network under the bilateral link formation initiated to Jackson and Watts (2002).

7 Robust stable network

In this section, we consider the robustness of stable network in the sense of Kajii and Morris (1997) and Ui (2001). By utilizing the method of Ui (2001), we will show that the unique network potential maximizer is robust.

Fix ϕ and consider the corresponding consent game $\Gamma_\phi = (\mathcal{A}, \pi_\phi)$. We introduce the notion of incomplete information perturbation of the underling game Γ_ϕ. Let T_i be a countable set of i's type and let $P \in \Delta(T)$ be a common prior over T. We assume $P(\{t_i\} \times T_{-i}) > 0$ for all $i \in N$ and $t_i \in T_i$. We denote $\mathcal{I} = ((T_i)_{i\in N}, P)$ as an information structure. A payoff function for player i is a bounded function $u_i : \mathbb{G}^N \times T \to \mathbb{R}$. Then, let us define $\pi_{u,i}(l, t) = u_i(g(l), t)$. We call $\tilde{\Gamma} = (\Gamma_u, \mathcal{I})$ as an incomplete information consent game.

A (mixed) strategy of player i is a function $\sigma_i : T_i \to \Delta(\mathcal{A}_i)$. Let Σ_i be the set of strategies of player i. We write $\sigma(l|t) = \Pi_{i\in N}\sigma_i(l_i|t_i)$ where $\sigma_i(l_i|t_i)$ is the probability of action l_i given $\sigma_i \in \Sigma_i$. Also, we write $\sigma_P(l) = \sum_{t\in T} P(t)\sigma(l|t)$.

[4]See Gilles and Chakrabarti (2007).

The payoff of strategy profile $\sigma \in \Sigma$ to player i is

$$U_i(\sigma) = \sum_{t \in T} \sum_{l \in \mathcal{A}} P(t)\sigma(l|t)\pi_{u,i}(l,t)P(t)$$

A strategy profile $\sigma \in \Sigma$ is a Bayesian Nash equilibrium of $\tilde{\Gamma}$ if, for each $i \in N$ and each $\sigma_i' \in \Sigma_i$, $U_i(\sigma) \geq U_i(\sigma_i', \sigma_{-i})$.

Following Ui (2001), we consider the next subset of T_i:

$$T_i^{u_i} = \{t_i \in T_i | u_i(g, (t_i, t_{-i})) = \phi(g) \text{ for all } g \in \mathbb{G}^N \text{ and } t_{-i} \in T_{-i} \text{ with } P(t_i, t_{-i}) > 0\}$$

We write $T^{\mathbf{u}} = \prod_{i \in N} T_i^{u_i}$.

Definition 4. An incomplete information consent game $\tilde{\Gamma}$ is an *ε-elaboration* of Γ_ϕ if $P(T^{\mathbf{u}}) = 1 - \varepsilon$ for all $\varepsilon \in [0, 1]$.

We want to consider what network configuration is robustly appeared in the real situation under the strategic behavior. To know this, we introduce robust stable network as follows.

Definition 5. A network g^* is *robust to all perturbations* if , for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \leq \bar{\varepsilon}$, every ε-perturbation of Γ_ϕ has a Bayesian Nash equilibrium σ with $\sigma_P^{\mathcal{N}}(g^*) \geq 1 - \delta$ where $\sigma_P^{\mathcal{N}}(g^*) = \sum_{l \in \mathcal{A}, g(l) = g^*} \sigma_P(l)$.

We call $t_i \in T_i \backslash T_i^{u_i}$ *committed type* if player i of type t_i has a unique maximizer $g^{t_i} \in \mathbb{G}^N$ with $u_i(g^{t_i}, (t_i, t_{-i})) > u_i(g, (t_i, t_{-i}))$ for all $g \in \mathbb{G}^N \backslash \{g^{t_i}\}$ and $t_{-i} \in T_{-i}$.

Definition 6. ε-elaboration of Γ_ϕ is *canonical* if every $t_i \in T_i \backslash T_i^{u_i}$ is committed type for all $i \in N$.

We next consider the slightly weaker version of robustness.[5]

Definition 7. A network g^* is *robust to canonical perturbations* if , for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \leq \bar{\varepsilon}$, every canonical ε-perturbation of Γ_ϕ has a Bayesian Nash equilibrium σ with $\sigma_P^{\mathcal{N}}(g^*) \geq 1 - \delta$ where $\sigma_P^{\mathcal{N}}(g^*) = \sum_{l \in \mathcal{A}, g(l) = g^*} \sigma_P(l)$.

The main result of Ui (2001) show that, under the potential game, the unique potential maximizer is robust to canonical elaboration. We state network analogue of this result in the next Theorem.

Theorem 6. *Let ϕ admits a network potential ω. Suppose that $\{g^*\} = \text{argmax}_{g \in \mathbb{G}^N} \omega(g)$. Then, g^* is robust to canonical perturbations.*

[5]The difference is remained open question in the literature.

Proof. Let us first define $V(\sigma) = \sum_{t\in T}\sum_{l\in\mathcal{A}} P(t)\sigma(l|t)\omega(g(l))P(t)$. By Theorem 4, $P(l) = \omega(g(l))$ is a potential function of consent game Γ_ϕ. Then, this V is an elaboration potential in the sense of Ui(2001). Note that, by definition of canonical elaboration, player i of $t_i \in T_i \backslash T_i^{u_i}$ has (weakly) dominant strategy, in particular, non-superfluous strategy for g^{t_i} i.e, $\hat{l}_{g^{t_i},i}$, which we denote $l_i^{t_i}$. Let us define $\Sigma_i^{\pi_{u,i}} = \{\sigma_i \in \Sigma | \sigma(l_i^{t_i}|t_i) = 1$ for all $t_i \in T_i \backslash T_i^{u_i}\}$. By Lemma 2 and 3 of Ui (2001), $\sigma^* \in \text{argmax}_{\sigma\in\Sigma^{\pi_u}} V(\sigma)$ is a Bayesian Nash equilibrium of $\tilde{\Gamma}$ and this exists.

Define $\omega^* \equiv \omega(g^*), \omega' \equiv \max_{g\in\mathbb{G}^N\backslash\{g^*\}}\omega(g)$, and $\omega'' \equiv \min_{g\in\mathbb{G}^N}\omega(g)$. Note that $\omega^* > \omega' \geq \omega''$. Let $\sigma \in \Sigma^u$ such that $\sigma_i(\hat{l}_{g^*,i}|t_i) = 1$ for all $i \in N$ and $t_i \in T_i^{u_i}$. Then, we have,

$$\begin{aligned} V(\sigma^*) \geq V(\sigma) =& \sum_{t\in T^{\mathbf{u}}}\sum_{l\in\mathcal{A}} \sigma(l|t)\omega(g(l))P(t) + \sum_{t\in T\backslash T^{\mathbf{u}}}\sum_{l\in\mathcal{A}} \sigma(l|t)\omega(g(l))P(t) \\ =& P(T^{\mathbf{u}})\omega(g^*) + \sum_{t\in T\backslash T^{\mathbf{u}}} P(t)\sum_{l\in\mathcal{A}} \sigma(l|t)\omega(g(l)) \\ \geq& P(T^{\mathbf{u}})\omega^* + [1 - P(T^{\mathbf{u}})]\omega'' = (1-\varepsilon)\omega^* + \varepsilon\omega''. \end{aligned}$$

We also have

$$\begin{aligned} V(\sigma^*) =& \sum_{l\in\mathcal{A}}[\sum_{t\in T} P(t)\sigma^*(l|t)]\omega(g(l)) \\ =& \sum_{l\in\mathcal{A}} \sigma_P^*(l)\omega(g(l)) \\ =& \sum_{g\in\mathbb{G}^N} \sigma_P^{*,\mathcal{N}}(g)\omega(g(l)) \\ =& \sigma_P^{*,\mathcal{N}}(g^*)\omega(g^*) + \sum_{g\in\mathbb{G}^N\backslash\{g^*\}} \omega(g) \\ \leq& \sigma_P^{*,\mathcal{N}}(g^*)\omega^* + [1 - \sigma_P^{*,\mathcal{N}}(g^*)]\omega'. \end{aligned}$$

Combining the above inequalities, we have

$$(1-\varepsilon)\omega^* + \varepsilon\omega'' \leq \sigma_P^{*,\mathcal{N}}(g^*)\omega^* + [1 - \sigma_P^{*,\mathcal{N}}(g^*)]\omega'.$$

and thus

$$\sigma_P^{*,\mathcal{N}}(g^*) \geq 1 - \epsilon\frac{\omega^* - \omega''}{\omega^* - \omega'}$$

For each $\delta > 0$, let $\bar{\varepsilon} = \delta(\omega^* - \omega')/(\omega^* - \omega'')$. Then, this implies that, for all $\varepsilon \leq \bar{\varepsilon}$, every canonical ε-elaboration has a Bayesian Nash equilibrium σ^* with $\sigma_P^{*,\mathcal{N}}(g^*) \geq 1 - \delta$, which completes that proof. □

The robustness is based on the equilibrium concept of dual strategic form game. In this sense, this result can be seen as the strategic foundation of pairwise stability. Note that unless the unique network potential maximizer is the complete network, the potential maximizer of corresponding consent game is not unique because all strategy profiles which induce the same network can be maximizer of it. Thus, the result of Ui (2001) cannot be directly applied to the consent game. Our robustness is defined in terms of network rather than in terms of Nash equilibrium of consent game directly. Thus, above problem can be evaded.

8 Conclusion

In this paper, we gave a full characterization for network potentials in terms of the Shapley value so that potentials in a TU game. Our result is network analogue to that of Ui (2000) in the game theory. Key technique is so called interaction potential method. By fully used this concept, we can also give much simpler proof for the equivalence result for the Myerson's consent game. By using the potential method, we give several results which are analogue to game theory literature such as uniqueness of pairwise network, stochastic stability, and robustness to incomplete information perturbation.

Recently, application of network formation models are widely considered. We believe potential method can be applied in such models and our result will be useful to identify whether a network potential function exists or not. This line of research will give new insight into the applied field.

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