Non-manipulable Agenda Setting and Voting *

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Abstract

When the set of alternatives includes so many elements, in a voting procedure, agenda setting is inevitable. This paper searches for a voting procedure that is not manipulated by agenda setters. Our purpose is designing a voting rule and determining who should be agenda setter. In particular, to control incentives of agenda setters, we design a way to provide monetary rewards to agenda setters. In the traditional model pioneered by Arrow (1951), we show that in the full domain, any unanimous social choice rule admits agenda manipulations. However, if we restrict a domain of social choice rules, there exists a voting procedure that avoids agenda manipulations. Moreover, in the model of Balinski and Laraki (2007), we find another kind of voting procedure that avoids agenda manipulations even in the full domain.

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1 Introduction

When the set of alternatives includes so many alternatives, we, in practice, set an agenda, that is, we select a subset of alternatives on which a vote is taken. Agenda setting is important because of deliberation, consideration, reduction of information, and so on. However, voting outcomes are significantly affected by an agenda set by agenda setters. In other words, there is a possibility that agenda setters can manipulate the outcome. This paper deals with this problem and searches for a voting procedure that is not manipulated by agenda setters.

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Agenda setter	Voter
$a_1 : yzx \cdots$	$1 : xzy \cdots$
$a_2 : zyx \cdots$	$2 : xzy \cdots$
	$3 : xyz \cdots$
	$4 : xyz \cdots$
	$5 : xyz \cdots$

Figure 1: Preferences of agenda setters and voters

The dangerousness of agenda manipulation is widely recognized since seminal works by McKelvey (1976, 1979). If an agenda is set by only one agenda setter, then obviously he/she can manipulate the outcome by selecting an agenda that is most favorable to him/her. To avoid such a problem, we consider a new agenda setting procedure, "Agenda setting in competition". This procedure is very simple. (i) We prepare at least two agenda setters. (ii) Each agenda setter proposes an alternative. (iii) An agenda is composed of these proposed alternatives. (iv) A vote is taken on this agenda.

However, even in this procedure, agenda manipulation is still possible. For example, suppose that there are so many alternatives $\{x, y, z, \ldots\}$, two agenda setters and five voters whose preferences are given by Figure 1. There, voter 1 ranks x at the top, z next, and so on. For this profile of voters, x is unanimously supported, and hence it is socially desirable. On the other hand, suppose that agenda setter a_1 proposes y and agenda setter a_2 proposes z. Then, an agenda is composed of $\{y, z\}$, and by majority voting on this agenda, y wins. However, this outcome is different from the socially desirable alternative x. Moreover, this pair of proposed alternatives (y, z) is a Nash equilibrium of a game between agenda setters, i.e., both of them can not benefit from proposing another alternative. [1]

The above difficulty occurs because preferences of agenda setters are depending on only alternatives. Accordingly, in this analysis, we control the incentives of agenda setters by using monetary rewards, i.e., we give agenda setters some amount of monetary rewards according to what they propose. In this case, agenda setters would have preferences depending also on an amount of monetary rewards they receive. Our purpose is designing a way to provide monetary rewards and designing a voting rule to make voting outcomes socially desirable.

We say that a social choice rule is *dominant strategy implementable* if there exists a voting rule by which alternatives chosen by the social choice rule

[1] For example, if agenda setter a_1 proposes alternative x instead of y, then by majority voting with respect to (x, z), x wins, but this is apparently unprofitable to a_1.

are elected at dominant strategy equilibria.[2] Similarly, we define *Nash implementable* social choice rules. In the traditional model pioneered by Arrow (1951), we show that if we assume the full domain of social choice rules, then any unanimous and dominant strategy implementable social choice rule is dictatorial. However, in a restricted domain, we guarantee the existence of a social choice rule that is dominant strategy implementable, Nash implementable and non-dictatorial. In the model following Balinski and Laraki (2007), we also show the existence of another social choice rule that is dominant strategy implementable and Nash implementable.

This paper is organized as follows. Section 2 examines the possibility of dominant strategy implementation and Nash implementation in the traditional model. Section 3 gives possibility results in the model following Balinski and Laraki (2007). Section 4 discusses who should be agenda setter. Section 5 concludes this study. All proofs are relegated to Appendix.

2 The model in line with Arrow (1951)

2.1 Definitions

Let X be the countable set of all possible *alternatives*, $A = \{a_1, a_2, \dots, a_m\}$ the finite set of *agenda setters* and $I = \{1, 2, \dots, |I|\}$ the finite set of *voters*. We assume that $|X| \geq 3$ and $|A| \geq 2$. For notational simplicity, we denote $X = \{x_1, x_2, \dots x_{|X|}\}$ or $X = \{x_1, x_2, \dots\}$.

In this section, we assume that each voter submits his ordering on X. An *ordering* is a complete and transitive binary relation $\succsim_i$ on X.[3] Let $\mathscr{R}$ be the set of orderings on X. An *ordering profile* of n voters is

$$\succsim = (\succsim_1, \succsim_2, \dots, \succsim_n) \in \mathscr{R}^n.$$

A domain is a subset of the set of ordering profiles $\mathscr{D} \subset \mathscr{R}^n$.

We assume that X includes so many alternatives. In this situation, we set an agenda, that is we select a subset of alternatives on which a vote is taken. We consider an agenda setting procedure in which each agenda setter $a_k \in A$ proposes an alternative $s_k \in X$, and each profile of proposed alternatives $s \equiv (s_1, s_2, \dots, s_m) \in X^m$ identifies an agenda $\{s_1, s_2, \dots, s_m\} \subset X$. We call this procedure "Agenda setting in competition". For each $s \in X^m$, denote a profile of proposed alternatives except a_k's by

$$s_{-k} \equiv (s_1, \dots, s_{k-1}, s_{k+1}, \dots, s_m) \in X^{m-1},$$

[2]We distinguish the terminology "social choice rule" from "voting rule". A social choice rule determines socially desirable alternatives, while a voting rule elects an alternative within an agenda.

[3] Completeness: for each $x, y \in X$, $x \succsim_i y$ or $y \succsim_i x$, Transitivity: for each $x, y, z \in X$, $x \succsim_i y$ and $y \succsim_i z$ together imply $x \succsim_i z$.

and for each $s'_k \in X$ let

$$(s'_k, s_{-k}) \equiv (s_1, \dots, s_{k-1}, s'_k, s_{k+1}, \dots, s_m).$$

For each $\succsim \in \mathscr{D}$ and $s \in X^m$, we denote the restriction of $\succsim$ into $\{s_1, s_2, \dots, s_m\}$ by $\succsim |_{\{s_1,s_2,\dots,s_m\}}$.

Definition 1. A *voting rule* is a function $f : \mathscr{D} \times X^m \to X$ such that

(i) for any $\succsim \in \mathscr{D}$ and any $s \in X^m$, $f(\succsim, s) \in \{s_1, s_2, \dots, s_m\}$,

(ii) for any $\succsim \in \mathscr{D}$ and any $s, s' \in X^m$, if $\{s_1, s_2, \dots, s_m\} = \{s'_1, s'_2, \dots, s'_m\}$, then $f(\succsim, s) = f(\succsim, s')$.

(iii) for any $\succsim, \succsim' \in \mathscr{D}$ and any $s \in X^m$,

$$\left[\forall i \in I,\ \succsim_i |_{\{s_1,s_2,\dots,s_m\}} = \succsim'_i |_{\{s_1,s_2,\dots,s_m\}}\right] \Longrightarrow f(\succsim, s) = f(\succsim', s).$$

Condition (i) requires that a voting rule elects an alternative from an agenda. Condition (ii) requires that a voting rule is independent of which agenda setter proposes an alternative. Condition (iii) requires that a voting rule is independent of voters' orderings on unproposed alternatives. Let $\mathscr{F}$ be the set of voting rules.

Each agenda setter $a_k \in A$ is concerned with a voting outcome (winner) and an amount of monetary reward he receives. We assume that each agenda setter $a_k \in A$ has a quasi-linear utility function $u_k : X \times \mathbb{R} \to \mathbb{R}$ such that for each $x \in X$ and each $r_k \in \mathbb{R}$, $u_k(x, r_k) = v_k(x) + r_k$. We call $v_k : X \to \mathbb{R}$ agenda setter a_k's valuation function. We assume that for each $a_k \in A$, there exists $\bar{r}_k \in \mathbb{R}$ such that $\sup_{x,y \in X}\{v_k(x) - v_k(y)\} < \bar{r}_k$. This condition says that there exists an amount of monetary reward $\bar{r}_k \in \mathbb{R}$ that makes $u_k(x, 0) < u_k(y, \bar{r}_k)$ for all $x, y \in X$.[4]

Each agenda setter receives a monetary reward according to a voting outcome, what he proposes and what other agenda setters propose. A reward function is a function $r_k : X \times X^m \to \mathbb{R}$ that maps each outcome of voting $x \in X$ and each $s \in X^m$ to an amount of monetary reward, $r_k(x, s) \in \mathbb{R}$, agenda setter a_k receives. Our purpose is designing a voting rule $f \in \mathscr{F}$ and reward functions $(r_1 \dots, r_m)$.

Let $c : X \times X^m \to \{1, 2, \dots, |A|\}$ be such that for each $x \in X$ and each $s \in X^m$,

$$c(x, s) \equiv |\{a_{k'} \in A : s_{k'} = x\}|.$$

[4]If X is a finite set, this condition is obviously satisfied. In Section 4, we address a case with some agenda setters having non-quasi-linear utilities.

$c(x,s) \in \{1,2,\dots,|A|\}$ represents the number of agenda setters who propose a voting outcome $x \in X$. In this paper, we particularly consider reward functions $(r_1.\dots,r_m)$ of the following form:

$$r_k(x,s) = \begin{cases} \frac{\bar{r}}{c(x,s)} & \text{if } \ s_k = x, \\ 0 \ \text{if } \ s_k \neq x, \end{cases}$$

where $\bar{r} \in \mathbb{R}_{++}$ is the total amount of monetary rewards.[5] We assume that $\bar{r} \in \mathbb{R}_{++}$ is sufficiently large to satisfy that for each $a_k \in A$ and each $x, y \in X$,

$$v_k(x) + \frac{\bar{r}}{m} > v_k(y).^6 \tag{1}$$

This condition implies that each agenda setter prefers to propose a winner rather than propose a loser.

For each $(f, \succsim) \in \mathscr{F} \times \mathscr{D}$, and each $a_k \in A$, $s_k \in X$ is a *dominant strategy* at $(f, \succsim)$ if for any $s_{-k} \in X^{m-1}$ and any $s'_k \in X$,

$$\begin{aligned} v_k(f(\succsim, s_k, s_{-k})) &+ r_k(f(\succsim, s_k, s_{-k}), s_k, s_{-k}) \\ &\geq v_k(f(\succsim, s'_k, s_{-k})) + r_k(f(\succsim, s'_k, s_{-k}), s'_k, s_{-k}). \end{aligned}$$

Let $\mathrm{DS}_k(f,\succsim)$ be the set of agenda setter a_k's dominant strategies at $(f,\succsim) \in \mathscr{F} \times \mathscr{D}$. Then, the set of dominant strategy equilibria at $(f,\succsim) \in \mathscr{F} \times \mathscr{D}$ is given by $\mathrm{DE}(f,\succsim) \equiv \prod_{k=1}^m \mathrm{DS}_k(f,\succsim)$.

2.2 A impossibility theorem

Although a voting rule is used to elect an alternative within an agenda, our primary purpose is to elect a socially desirable alternative in X that is supposed to be chosen if we could know a whole ordering profile on X. A *social choice rule* is a correspondence $F : \mathscr{D} \twoheadrightarrow X$ that maps each ordering profile $\succsim \in \mathscr{D}$ to a nonempty subset $F(\succsim) \subset X$. We assume that for any $\succsim \in \mathscr{D}$, $F(\succsim)$ selects socially desirable alternatives.

We are interested in "implementable" social choice rules. For each $f \in \mathscr{F}$ and each $\succsim \in \mathscr{D}$, let

$$f(\succsim, \mathrm{DE}(f,\succsim)) \equiv \{x \in X : f(\succsim, s) = x \text{ for some } s \in \mathrm{DE}(f,\succsim)\}.$$

Definition 2. A social choice rule $F : \mathscr{D} \twoheadrightarrow X$ is *dominant strategy implementable* if there exists a voting rule $F \in \mathscr{F}$ such that for any $\succsim \in \mathscr{D}$,

$$F(\succsim) = f(\succsim, \mathrm{DE}(f,\succsim)).$$

[5] We interpret $\bar{r}$ as a total cost for avoiding agenda manipulations.

Note that this notion of implementation is a bit different from Maskin (1977, 1999)'s. In Maskin (1977, 1999)'s theorem, players of a game have preferences depending only on alternatives, while in this paper players of a game, agenda setters, have preferences depending also on monetary rewards. Unanimity requires that if there is an alternative which is the best for all voters, then a social choice rule chooses only this alternative.

Unanimity. A social choice rule $F : \mathscr{D} \twoheadrightarrow X$ is *unanimous* if for any $x \in X$ and any $\succsim \in \mathscr{D}$,

$$[\forall i \in I,\ \forall y \in X \setminus \{x\},\ x \succ_i y\,] \Longrightarrow F(\succsim) = \{x\}.$$

A social choice rule $F : \mathscr{D} \twoheadrightarrow X$ is *dictatorial* if there exists $i \in I$ such that for any $x \in X$ and any $\succsim \in \mathscr{D}$, whenever $x \succ_i y$ for any $y \in X \setminus \{x\}$, $F(\succsim) = \{x\}$.

Theorems 1 implies a difficulty of the dominant strategy implementation.

Theorem 1. *Suppose $\mathscr{D} = \mathscr{R}^n$. If a social choice rule $F : \mathscr{R}^n \twoheadrightarrow X$ is unanimous and dominant strategy implementable, then F is dictatorial.*

2.3 Restricting a domain

In Section 2.2, we show that if we assume the full domain, i.e., $\mathscr{D} = \mathscr{R}^n$, then the dominant strategy implementation is impossible. In this section, we search for a domain that avoids this impossibility.

For each voting rule $f \in \mathscr{F}$, $x \in X$ is a *f-Condorcet winner* at $\succsim \in \mathscr{D}$, if for any $y \in X$ and any $s \in X^m$,

$$[\{s_1, \ldots, s_m\} = \{x, y\}] \Longrightarrow f(\succsim, s) = x.$$[7]

A domain $\mathscr{D} \subset \mathscr{R}^n$ *admits a f-Condorcet winner* if for any $\succsim \in \mathscr{D}$, there exists a f-Condorcet winner $x \in X$ at $\succsim$. The f-Condorcet rule is a social choice rule $F_f : \mathscr{D} \twoheadrightarrow X$ such that for any $\succsim \in \mathscr{D}$,

$$F_f(\succsim) = \{x \in X : x \text{ is a } f\text{-Condorcet winner at } \succsim\}.$$

The f-Condorcet rule is well defined if a domain $\mathscr{D} \subset \mathscr{R}^n$ admits a f-Condorcet winner.

Theorem 2 states that if agenda setters' valuation functions are constant, then by restricting a domain the f-Condorcet rule becomes dominant strategy implementable.

[7] A Condorcet winner in its natural form is defined by a voting rule that compares pairs of alternatives by a simple majority rule.

Theorem 2. *Suppose that a domain $\mathscr{D} \subset \mathscr{R}^n$ admits a f-Condorcet winner. Suppose also that for any agenda setter $a_k \in A$ and any $x, y \in X$, $v_k(x) = v_k(y)$. Then, the f-Condorcet rule F_f is dominant strategy implementable.*

If agenda setters' valuation functions are not constant, Theorem 2 does not hold. However, even in such a case, the f-Condorcet rule satisfies another notion of implementation. For each $(f, \succsim) \in \mathscr{F} \times \mathscr{D}$, $s \in X^m$ is a *Nash equilibrium* at $(f, \succsim)$ if for any $a_k \in A$, and any $s'_k \in X$,

$$\begin{aligned} v_k(f(\succsim, s_k, s_{-k}))+&r_k(f(\succsim, s_k, s_{-k}), s_k, s_{-k}) \\ &\geq v_k(f(\succsim, s'_k, s_{-k})) + r_k(f(\succsim, s'_k, s_{-k}), s'_k, s_{-k}). \end{aligned}$$

Let $\mathrm{NE}(f, \succsim)$ be the set of Nash equilibria at $(f, \succsim) \in \mathscr{F} \times \mathscr{D}$.

Definition 3. A social choice rule $F : \mathscr{D} \twoheadrightarrow X$ is Nash implementable if there exists a voting rule $f \in \mathscr{F}$ such that for any $\succsim \in \mathscr{D}$,

$$F(\succsim) = f(\succsim, \mathrm{NE}(f, \succsim)).$$

Theorem 3 states that in a restricted domain, the f-Condorcet rule becomes Nash implementable.

Theorem 3. *Suppose that a domain $\mathscr{D} \subset \mathscr{R}^n$ admits a f-Condorcet winner. Then the f-Condorcet rule F_f is Nash implementable.*

3 The model in line with Balinski and Laraki (2007)

3.1 Definitions

In this section, we follow the model of Balinski and Laraki (2007, 2010). Here, each voter submits his grade of alternatives rather than his ordering on X. A *common language* is a finite set of grades $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{|\Lambda|}\}$. For example, $\Lambda = \{\text{Excellent}, \text{Very Good}, \text{Good}, \text{Bad}, \text{Very Bad}\}$. Λ is ordered by the complete, transitive, and anti-symmetric binary relation $\geqslant$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_{|\Lambda|}$.[8]

For each alternative $x \in X$, each voter $i \in I$ submits his/her grade of x, $L(i, x) \in \Lambda$. A *grade profile* is a $|I| \times |X|$-matrix $L \in \Lambda^{I \times X}$.

As with Section 2, each profile of alternatives $s = (s_1, s_2, \ldots, s_m) \in X^m$ identifies an agenda $\{s_1, s_2, \ldots, s_m\} \subset X$, where each $s_k \in X$ is an alternative proposed by agenda setter $a_k \in A$.

[8] Anti-symmetry: for each $\lambda, \lambda' \in \Lambda$, $\lambda \geqslant \lambda'$ and $\lambda' \geqslant \lambda$ implies $\lambda = \lambda'$.

Definition 4. A *voting rule* is a function $f : \Lambda^{I\times X} \times X^m \to X$ such that

(i) for any $L \in \Lambda^{I\times X}$ and any $s \in X^m$, $f(L, s) \in \{s_1, s_2, \ldots, s_m\}$,

(ii) for any $L \in \Lambda^{I\times X}$ and any $s, s' \in X^m$, if $\{s_1, s_2, \ldots, s_m\} = \{s'_1, s'_2, \ldots, s'_m\}$, then $f(L, s) = f(L, s')$.

(iii) for any $L, L' \in \Lambda^{I\times X}$ and any $s \in X^m$,

$$[\, \forall i \in I, \forall x \in \{s_1, s_2, \ldots, s_m\}, L(i, x) = L'(i, x)\,] \Longrightarrow f(L, s) = f(L', s).$$

Let $\mathscr{F}$ be the set of voting rules.[9] Each agenda setter $a_k \in A$ has the same utility function as in Section 2. Definitions and assumptions on reward functions $(r_1, \ldots, r_m)$ are also identical to that of Section 2.

For each $(f, L) \in \mathscr{F} \times \Lambda^{I\times X}$, and each $a_k \in A$, $s_k \in X$ is a *dominant strategy* at (f, L) if for any $s_{-k} \in X^{m-1}$ and any $s'_k \in X$,

$$\begin{aligned} v_k(f(L, s_k, s_{-k})) + & r_k(f(L, s_k, s_{-k}), s_k, s_{-k}) \\ & \geq v_k(f(L, s'_k, s_{-k})) + r_k(f(L, s'_k, s_{-k}), s'_k, s_{-k}). \end{aligned}$$

Let $\mathrm{DS}_k(f, L)$ be the set of agenda setter a_k's dominant strategies at $(f, L) \in \mathscr{F} \times \Lambda^{I\times X}$. Then, the set of dominant strategy equilibria at $(f, L) \in \mathscr{F} \times \Lambda^{I\times X}$ is given by $\mathrm{DE}(f, L) \equiv \prod_{k=1}^{m} \mathrm{DS}_k(f, L)$.

For each $(f, L) \in \mathscr{F} \times \Lambda^{I\times X}$, $s \in X^m$ is a *Nash equilibrium* at (f, L) if for any $a_k \in A$ and any $s'_k \in X$,

$$\begin{aligned} v_k(f(L, s_k, s_{-k})) + & r_k(f(L, s_k, s_{-k}), s_k, s_{-k}) \\ & \geq v_k(f(L, s'_k, s_{-k})) + r_k(f(L, s'_k, s_{-k}), s'_k, s_{-k}). \end{aligned}$$

Let $\mathrm{NE}(f, L)$ be the set of Nash equilibria at $(f, L) \in \mathscr{F} \times \Lambda^{I\times X}$.

3.2 Possibility Theorems

A *social choice rule* is a correspondence $F : \Lambda^{I\times X} \twoheadrightarrow X$ that maps each grade profile $L \in \Lambda^{I\times X}$ to a nonempty subset $F(L) \subset X$.[10] Again, for each $L \in \Lambda^{I\times X}$, $F(L) \subset X$ is a set of socially desirable alternatives. For each $f \in \mathscr{F}$ and each $L \in \Lambda^{I\times X}$, let

$$f(L, \mathrm{DE}(f, L)) \equiv \{x \in X : f(L, s) = x \text{ for some } s \in \mathrm{DE}(f, L)\}.$$

[9] Note that the domain of voting rules in this section is different from those in Section 2.

[10] Note that the domain of social choice rules in this section is different from those in Section 2.

Definition 5. A social choice rule $F : \Lambda^{I\times X} \twoheadrightarrow X$ is dominant strategy implementable if there exists a voting rule $f \in \mathscr{F}$ such that for any $L \in \Lambda^{I\times X}$,

$$F(L) = f(L, \mathrm{DE}(f, L)).$$

For each $f \in \mathscr{F}$ and each $L \in \Lambda^{I\times X}$, let

$$f(L, \mathrm{NE}(f, L)) \equiv \{x \in X : f(L, s) = x \text{ for some } s \in \mathrm{NE}(f, L)\}.$$

Definition 6. A social choice rule $F : \Lambda^{I\times X} \twoheadrightarrow X$ is Nash implementable if there exists a voting rule $f \in \mathscr{F}$ such that for any $L \in \Lambda^{I\times X}$,

$$F(L) = f(L, \mathrm{NE}(f, L)).$$

The following is an example of a social choice rule.

The median rule.
For each $L \in \Lambda^{I\times X}$ and $x \in X$, the *median sequence* of L at x, denoted by $\{\mu_h^L(x)\}_{h=1}^n \in \Lambda^n$, is defined by as follows: Let

$$\mu_1^L(x) \equiv \operatorname*{med}_{i\in I} L(i, x),$$

where $\operatorname*{med}_{i\in I} L(i, x)$ denotes the median of $L(\cdot, x)$.[11] Take some $i_1 \in I$ such that $L(i_1, x) = \mu_1^L(x)$. Now, let

$$\mu_2^L(x) \equiv \operatorname*{med}_{i\in I\setminus\{i_1\}} L(i, x).$$

Take some $i_2 \in I$ such that $L(i_2, x) = \mu_2^L(x)$. Similarly, for each $h \in \{1, 2, \dots, n\}$ define

$$\mu_h^L(x) \equiv \operatorname*{med}_{i\in I\setminus\{i_1\}\setminus\cdots\setminus\{i_{h-1}\}} L(i, x).$$

The median ranking is a function $R_m : \Lambda^{I\times X} \to \mathscr{R}$ such that for any $L \in \Lambda^{I\times X}$ and any $x_j, x_{j'} \in X$, $x_j \, R_m(L) \, x_{j'}$ if and only if one of the following two conditions is satisfied[12][13]:

[11] To be precise, a median of $L(\cdot, x)$ is a grade $\lambda \in \Lambda$ such that

$$|\{i \in I : L(i, x) \leqslant \lambda\}| \geq \frac{|I|}{2} \quad \text{and} \quad |\{i \in I : \lambda \geqslant L(i, x)\}| \geq \frac{|I|}{2}.$$

There exist at most two medians. When there are two medians, following Balinski and Laraki (2007), we throughout take the little median, that is, we take $\lambda \in \Lambda$ such that

$$|\{i \in I : L(i, x) \geqslant \lambda\}| \geq \frac{2|I| + 2}{2} \quad \text{and} \quad |\{i \in I : \lambda \geqslant L(i, x)\}| \geq \frac{|I|}{2}.$$

[12] Note that in this section all alternatives in X are numbered, i.e., X is denoted by $X = \{x_1, x_2, \dots\}$ or $X = \{x_1, x_2, \dots x_{|X|}\}$.

[13] In Balinski and Laraki (2007), this ranking is called majority ranking.

(i) There exists $h \in \{1, 2, \ldots, n\}$ such that $\mu_h^L(x_j) > \mu_h^L(x_{j'})$ and for all $h' < h$, $\mu_{h'}^L(x_j) = \mu_{h'}^L(x_{j'})$.

(ii) For all $h \in \{1, 2, \ldots, n\}$, $\mu_h^L(x_j) = \mu_h^L(x_{j'})$ and $j \leq j'$.

Note that for any $L \in \Lambda^{I \times X}$, $R_m(L)$ is complete, transitive and anti-symmetric. Then, the median rule $F_m : \Lambda^{I \times X} \twoheadrightarrow X$ is defined by

$$F_m(L) = \{x \in X : x \, R_m(L) \, y \ \text{ for all } y \in X\}.$$

Clearly, for all $L \in \Lambda^{I \times X}$, $F_m(L) \neq \emptyset$. Hence F_m is a social choice rule. We obtain two kinds of possibility theorems.

Theorem 4. *Suppose that for any agenda setter $a_k \in A$ and any $x, y \in X$, $v_k(x) = v_k(y)$. Then, the median rule F_m is dominant strategy implementable.*

Theorem 5. *The median rule F_m is Nash implementable.*

4 Discussion

4.1 Who should be agenda setter?

In Section 2 and Section 3, we assume that each agenda setter has a quasi-linear utility. However, in reality there would be a case where this assumption does not hold. For example, in 2015, Osaka, Japan, a referendum on the implementation of the so-called *Osaka Metropolis plan* was held. Osaka Metropolis plan is a plan to merge Osaka city into *Osakafu* by splitting the city into five special wards. This plan was proposed by Osaka city mayor *Toru Hashimoto*, and hence he was an practical agenda setter of this referendum. Since Osaka Metropolis plan was a main goal of his political activity, he may not have a quasi-linear utility. In such a case, any possibility theorems in Section 2 and Section 3 do not hold in general. Furthermore, he would have a very strong power as we have seen in Section 1.

To avoid this problem, we assert that agenda setting must be left to third party organization. In other words, we claim establishment of organizations who have quasi-linear utilities as in Section 2 and Section 3. If we establish such organizations, then agenda manipulations can be avoided in a sense of possibility results in Section 2.3 and Section 3.2. Moreover, if valuation functions of these established organizations are close to constant, then the cost for avoiding the manipulations, the total amount of monetary rewards, becomes pretty small.

4.2 Non-quasi-linear utilities

Even if we establish third party agenda setters, there would be a case where some other agenda setters have non-quasi-linear utility functions. For the example of Osaka Metropolis plan, Toru Hashimoto may have to be one of the agenda setters because of a real-life constraint. This section deals with such cases.

Let $A_1, A_2 \subset A$ be partition of A, i.e., $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$. In our setting, A_1 is composed of third party organizations and A_2 is composed of others. We assume that each agenda setter $a_k \in A_1$ has a quasi-linear utility function $u_k : X \times \mathbb{R} \to \mathbb{R}$ such that for any $x \in X$ and any $r_k \in \mathbb{R}$, $u_k(x, r_k) = v_k(x) + r_k$. As with Section 2 and Section 3, we assume that for each $a_k \in A_1$, there exists $\bar{r}_k \in \mathbb{R}$ such that $\sup_{x,y \in X}\{v_k(x) - v_k(y)\} < \bar{r}_k$. On the other hand, each agenda setter $a_k \in A_2$ has a non-quasi-linear utility function $u_k : X \times \mathbb{R} \to \mathbb{R}$.

In this section, for each $a_k \in A$, we consider another kind of reward function $r'_k : X \times X^m \to \mathbb{R}$. Let $c' : X \times X^m \to \{1, 2, \ldots, |A|\}$ be such that for each $x \in X$ and each $s \in X^m$,

$$c'(x, s) \equiv |\{a_{k'} \in A_1 : s_{k'} = x\}|.$$

$c'(x, s) \in \{1, 2, \ldots, |A_1|\}$ counts the number of agenda setters in A_1 who proposes a voting outcome $x \in X$. For each voting outcome $x \in X$ and each $s \in X^m$, agenda setter $a_k \in A_1$ receives a monetary reward $r'_k(x, s) \in \mathbb{R}$ such that

$$r'_k(x, s) = \begin{cases} \frac{\bar{r}'}{c'(x,s)} & \text{if } \; s_k = x, \\ 0 \;\; \text{if } \; s_k \neq x, \end{cases}$$

where $\bar{r}' \in \mathbb{R}_{++}$ is a total amount of monetary rewards. On the other hand, each agenda setter $a_k \in A_2$ receives a monetary reward $r'_k(x, s) = 0$ for any $x \in X$ and any $s \in X^m$.

We here address the model of Section 3, that is, each voter $i \in I$ submits his grades $L(\cdot, i) \in \Lambda^{|X|}$ of alternatives. [14] Definitions of voting rule and others are the same as those of Section 3. We obtain the following results.

For each $f \in \mathscr{F}$, each $L \in \Lambda^{I \times X}$ and each $s_{-k} \in X^m$, let

$$f(L, \mathrm{DS}_k(f, L), s_{-k}) \equiv \{x \in X : f(L, s_k, s_{-k}) = x \text{ for some } s_k \in \mathrm{DS}_k(f, L)\}.$$

Proposition 1 states that if there is an agenda setter who has a constant valuation function, then voting outcomes are desirable independently of alternatives proposed by other agenda setters.

[14] By restricting a domain, we can show the similar results even in the model of Section 2.

Proposition 1. *Suppose that there exists an agenda setter $a_k \in A_1$ such that for any $x, y \in X$, $v_k(x) = v_k(y)$. Then, there exists a voting rule $f \in \mathscr{F}$ such that for any $L \in \Lambda^{I\times X}$ and any $s_{-k} \in X^{m-1}$,*

$$f(L, \mathrm{DS}_k(f, L), s_{-k}) = F_m(L),$$

where F_m is the median rule.

Proposition 2 states that if there are at least two agenda setters with quasi-linear utilities, then voting outcomes are desirable in Nash equilibria.

Proposition 2. *Suppose $|A_1| \geq 2$. If the total amount of monetary rewards $\bar{r} \in \mathbb{R}_{++}$ is sufficiently large, then the median rule F_m is Nash implementable.*

5 Conclusion

This paper examined a new agenda setting procedure, "Agenda setting in competition". We designed reward functions and voting rules. This paper showed that agenda manipulation can be avoided if we restrict a domain of social choice rules or each voter submits grades of alternatives rather than an ordering. It is an interesting future topic if with some other reward functions and voting rules, agenda manipulations can be avoided.

Appendix

Proof of Theorem 1

Suppose that $f \in \mathscr{F}$ implements a unanimous social choice rule $F : \mathscr{R}^n \twoheadrightarrow X$ in dominant strategy equilibria. For each $x, y \in X$, let

$$S^{xy} \equiv \{s \in X^m : \{s_1, \dots, s_m\} = \{x, y\}\}.$$

Lemma 1. *For any $a_k \in A$, any $\succsim \in \mathscr{R}^n$ and any $x \in X$, if $x \in \mathrm{DS}_k(f, \succsim)$, then for any $y \in X$ and any $s^{xy} \in S^{xy}$, $f(\succsim, s^{xy}) = x$.*

Proof. Take any $a_k \in A$, any $\succsim \in \mathscr{R}^n$ and any $x \in X$. Suppose that $x \in \mathrm{DS}_k(f, \succsim)$. Suppose, by contradiction, that there exists $y \in X$ and $s^{xy} \in S^{xy}$ such that $f(\succsim, s^{xy}) = y \neq x$. Let $s_{-k} \in X^{m-1}$ be such that $s_{-k} = (\underbrace{y, \dots, y}_{m-1})$. By definition (ii) of voting rule, $f(\succsim, x, s_{-k}) = f(\succsim, s^{xy}) = y$. Moreover, by definition (i) of voting rule, $f(\succsim, y, s_{-k}) = y$. Then,

$$\begin{aligned} v_k(f(\succsim, y, s_{-k})) + r_k(f(\succsim, y, s_{-k}), y, s_{-k}) &= v_k(y) + \frac{\bar{r}}{m} \\ &> v_k(y) + 0 = v_k(f(\succsim, x, s_{-k})) + r_k(f(\succsim, x, s_{-k}), x, s_{-k}). \end{aligned}$$

Hence $x \notin \mathrm{DS}_k(f, \succsim)$, a contradiction. $\square$

Lemma 2. *For any $x, y \in X$, any $\succsim \in \mathscr{R}^n$ and any $s^{xy} \in S^{xy}$,*

$$[\, x \succ_i y \quad \forall i \in I \,] \Longrightarrow f(\succsim, s^{xy}) = x.$$

Proof. Take any $x, y \in X$, any $\succsim \in \mathscr{R}^n$ and any $s^{xy} \in S^{xy}$. Suppose that $x \succ_i y$ for any $i \in I$. Define an ordering profile $\succsim' \in \mathscr{R}^n$ to satisfy

$$x \succ_i' z \quad \forall i \in I, \quad \forall z \in X \setminus \{x\}.$$

By definition of dominant strategy implementation, and Unanimity of F,

$$f(\succsim', \mathrm{DE}(\succsim', f)) = F(\succsim') = \{x\}. \tag{2}$$

Suppose, by contradiction, that $f(\succsim', s^{xy}) \neq x$. Then, by Lemma 1, for any $a_k \in A$, $x \notin \mathrm{DS}_k(f, \succsim')$. Hence $x \notin f(\succsim', \mathrm{DE}(\succsim', f))$, a contradiction to (2). Thus, $f(\succsim', s^{xy}) = x$. Then, by definition (iii) of voting rule f, $f(\succsim, s^{xy}) = x$. $\square$

Lemma 3. *For any $\succsim \in \mathscr{R}^n$, $x, y, z \in X$, $s^{xy} \in S^{xy}$, $s^{yz} \in S^{yz}$ and $s^{xz} \in S^{xz}$,*

$$[\, f(\succsim, s^{xy}) = x \quad \textit{and} \quad f(\succsim, s^{yz}) = y \,] \Longrightarrow f(\succsim, s^{xz}) = x.$$

Proof. Take any $\succsim \in \mathscr{R}^n$, $x, y, z \in X$, $s^{xy} \in S^{xy}$, $s^{yz} \in S^{yz}$ and $s^{xz} \in S^{xz}$. Suppose that $f(\succsim, s^{xy}) = x$ and $f(\succsim, s^{yz}) = y$. Let $\succsim' \in \mathscr{R}^n$ be such that for each $i \in I$,

(i) $\succsim_i' |_{\{x,y,z\}} \equiv \succsim_i |_{\{x,y,z\}}$,

(ii) $v \succ_i' w$, $\forall v \in \{x, y, z\}$, $\forall w \in X \setminus \{x, y, z\}$.

Then, by definition (iii) of f, $f(\succsim', s^{xy}) = x$ and $f(\succsim', s^{yz}) = y$. Suppose, by contradiction, that $f(\succsim', s^{xz}) = z$. Let us show that for any $v \in X$, there exists $w \in X \setminus \{v\}$ and $s^{vw} \in S^{vw}$ such that $f(\succsim', s^{vw}) = w$. Take any $v \in X$. At first, suppose that $v \in \{x, y, z\}$. Since $f(\succsim', s^{xy}) = x$, $f(\succsim', s^{yz}) = y$ and $f(\succsim', s^{xz}) = z$, there exists $w \in \{x, y, z\}$ such that $f(\succsim', s^{vw}) = w$.

Next, suppose that $v \in X \setminus \{x, y, z\}$. By definition of $\succsim'$, $x \succ_i' v$ for all $i \in I$. Then, by Lemma 2, for any $s^{xv} \in S^{xv}$, $f(\succsim', s^{xv}) = x$.

Then, by Lemma 1, for any $v \in X$ and any $a_k \in A$, $v \notin \mathrm{DS}_k(\succsim', f)$, i.e., $\mathrm{DS}_k(\succsim', f) = \emptyset$. Hence $\mathrm{DE}(\succsim', f) = \emptyset$. Then, $F(\succsim') = f(\succsim', \mathrm{DE}(\succsim', f)) = \emptyset$, a contradiction. Therefore, $f(\succsim', s^{xz}) = x$. Then, by definition (iii) of f, $f(\succsim, s^{xz}) = x$. $\square$

Lemma 4. *There exists $i \in I$ such that for any $x, y \in X$ and any $\succsim \in \mathscr{R}^n$,*

$$[\, x \succ_i y \,] \Longrightarrow \ f(\succsim, s^{xy}) = x \quad \forall s^{xy} \in S^{xy}.$$

Proof. Let $R : \mathscr{R}^n \to \mathscr{R}$ be such that for any $x, y \in X$, $x\, R(\succsim)\, y$ if for some $s^{xy} \in S^{xy}$, $f(\succsim, s^{xy}) = x$. Let us show that we can apply Arrow's impossibility theorem to R.

(i) Completeness. Take any $\succsim \in \mathscr{R}^n$, any $x, y \in X$ and any $s^{xy} \in S^{xy}$. At first, let us consider the case $f(\succsim, s^{xy}) = x$. Then, by definition of R, $x\, R(\succsim)\, y$. Next, let us consider the case $f(\succsim, s^{xy}) = y$. Then, by definition of R, $y\, R(\succsim)\, x$. Therefore, R is complete.

(ii) Transitivity. Take any $\succsim \in \mathscr{R}^n$ and any $x, y, z \in X$. Suppose that $x\, R(\succsim)\, y$ and $y\, R(\succsim)\, z$. Then, there exists $s^{xy} \in S^{xy}$ such that $f(\succsim, s^{xy}) = x$, and $s^{yz} \in S^{yz}$ such that $f(\succsim, s^{yz}) = y$. Take some $s^{xz} \in S^{xz}$. Then, by Lemma 3, $f(\succsim, s^{xz}) = x$. Therefore, $x\, R(\succsim)\, z$.

(iii) Unanimity. Take any $\succsim \in \mathscr{R}^n$, any $x \in X$ and any $y \in X \setminus \{x\}$. Suppose that for any $i \in I$, $x \succ_i y$. Let us show that $x\, R(\succsim)\, y$ and not $y\, R(\succsim)\, x$. Take any $s^{xy} \in S^{xy}$. Then, by Lemma 2, $f(\succsim, s^{xy}) = x$. Therefore, $x\, R(\succsim)\, y$ and not $y\, R(\succsim)\, x$.

(iv) Binary independence. Take any $x, y \in X$ and any $\succsim, \succsim' \in \mathscr{R}^n$ with $\succsim |_{\{x,y\}} = \succsim' |_{\{x,y\}}$. Let us show that $x\, R(\succsim)\, y$ if and only if $x\, R(\succsim')\, y$. We first show "only if" part. Suppose that $x\, R(\succsim)\, y$. By definition of R, there exists $s^{xy} \in S^{xy}$ such that $f(\succsim, s^{xy}) = x$. Then, by definition (iii) of f, $f(\succsim', s^{xy}) = x$. Therefore, $x\, R(\succsim')\, y$. A similar argument shows "if" part.

Then, we can apply Arrow's impossibility theorem to R, that is, there exists $i \in I$ such that for any $x, y \in X$ and any $\succsim \in \mathscr{R}^n$, $x \succ_i y$ implies $x\, R(\succsim)\, y$ and not $y\, R(\succsim)\, x$.

Now, take any $x, y \in X$ and any $\succsim \in \mathscr{R}^n$. Suppose that $x \succ_i y$. Since $x\, R(\succsim)\, y$, there exists $s^{xy} \in S^{xy}$ such that $f(\succsim, s^{xy}) = x$. Then, by definition (ii) of f, $f(\succsim, s) = x$ for any $s \in S^{xy}$. $\square$

Take any $x \in X$ and any $\succsim \in \mathscr{R}^n$ such that $x \succ_i y$ for all $y \in X \setminus \{x\}$. Then, by Lemma 4, for any $y \in X \setminus \{x\}$ and any $s^{xy} \in S^{xy}$, $f(\succsim, s^{xy}) = x$. Hence by Lemma 1, for any $a_k \in A$ and $y \in X \setminus \{x\}$, $y \notin \mathrm{DS}_k(f, \succsim)$. Therefore, for any $y \in X \setminus \{x\}$, $y \notin f(\succsim, \mathrm{DE}(f, \succsim)) = F(\succsim)$. If $x \notin F(\succsim)$, then $F(\succsim) = \emptyset$, a contradiction to $F(\succsim) \neq \emptyset$. Hence $F(\succsim) = \{x\}$. $\square$

Proof of Theorem 2

Suppose that a domain $\mathscr{D} \subset \mathscr{R}^n$ admits a f-Condorcet winner. Suppose also that for any agenda setter $a_k \in A$ and any $x, y \in X$, $v_k(x) = v_k(y)$. Then, for any $\succsim \in \mathscr{D}$, there exists unique $x \in X$ such that for any $y \in X$ and any $s \in X^m$ with $\{s_1, \ldots, s_m\} = \{x, y\}$, $f(\succsim, s) = x$. Define a voting rule $f' \in \mathscr{F}$ to satisfy that for any $\succsim \in \mathscr{D}$ and any $s \in X^m$, $x \in \{s_1, s_2, \ldots, s_m\}$ implies $f'(\succsim, s) = x$.

Let us show that for any $\succsim \in \mathscr{D}$, $f'(\succsim, \mathrm{DE}(\succsim, f')) = F_f(\succsim)$. Take any $\succsim \in \mathscr{D}$. We shall show that for any $a_k \in A$, $\mathrm{DS}_k(\succsim, f') = F_f(\succsim)$. Take any $a_k \in A$. At first, note that $F_f(\succsim) = \{x\}$. We first show that $x \in \mathrm{DS}_k(\succsim, f')$. Take any $s_{-k} \in X^{m-1}$ and $s'_k \in X$. Since $x \in \{s_1, \ldots, s_{k-1}, x, s_{k+1}, \ldots, s_m\}$, $f'(\succsim, x, s_{-k}) = x$. Suppose that $c(f'(\succsim, x, s_{-k}), x, s_{-k}) = 1$. Since for any $x, y \in X$, $v_k(x) = v_k(y)$,

$$\begin{aligned}
u_k(f'(\succsim, x, s_{-k}), x, s_{-k}) &= v_k(x) + \bar{r} \\
&\geq v_k(x) + r_k(f'(\succsim, s'_k, s_{-k}), s'_k, s_{-k}) \\
&= v_k(f'(\succsim, s'_k, s_{-k})) + r_k(f'(\succsim, s'_k, s_{-k}), s'_k, s_{-k}) \\
&= u_k(f'(\succsim, s'_k, s_{-k}), s'_k, s_{-k}).
\end{aligned}$$

Next, suppose that $c(f'(\succsim, x, s_{-k}), x, s_{-k}) > 1$. If $s'_k = x$, then clearly

$$u_k(f'(\succsim, x, s_{-k}), x, s_{-k}) = u_k(f'(\succsim, s'_k, s_{-k}), s'_k, s_{-k}).$$

Therefore, we suppose that $s'_k \neq x$. Since $c(f'(\succsim, x, s_{-k}), x, s_{-k}) > 1$, there exists $a_\ell \in A$ with $a_\ell \neq a_k$ such that $s_\ell = x$. Then, $x \in \{s_1, \ldots, s_{k-1}, s'_k, s_{k+1}, \ldots, s_m\}$. Hence $f'(\succsim, s'_k, s_{-k}) = x$. Therefore, since $x = f'(\succsim, x, s_{-k})$ and $s'_k \neq x = f'(\succsim, s'_k, s_{-k})$,

$$\begin{aligned}
u_k(f'(\succsim, x, s_{-k}), x, s_{-k}) &= v_k(x) + \frac{\bar{r}}{c(x, x, s_{-k})} \\
&> v_k(x) + 0 \\
&= v_k(f'(\succsim, s'_k, s_{-k})) + r_k(f'(\succsim, s'_k, s_{-k}), s'_k, s_{-k}) \\
&= u_k(f'(\succsim, s'_k, s_{-k}), s'_k, s_{-k}).
\end{aligned}$$

Thus, $x \in \mathrm{DS}_k(\succsim, f')$.

We next show that for any $y \in X \setminus \{x\}$, $y \notin \mathrm{DS}_k(\succsim, f')$. Take any $y \in X \setminus \{x\}$. Let $s_{-k} \in X^{m-1}$ be such that $s_{-k} = (\underbrace{x, \ldots, x}_{m-1})$. Then, $x = f'(\succsim, x, s_{-k})$ and $y \neq x = f'(\succsim, y, s_{-k})$. Hence

$$\begin{aligned}
u_k(f'(\succsim, x, s_{-k}), x, s_{-k}) &= v_k(x) + \frac{\bar{r}}{m} \\
&> v_k(x) + 0 \\
&= v_k(f'(\succsim, y, s_{-k})) + r_k(f'(\succsim, y, s_{-k}), y, s_{-k}) \\
&= u_k(f'(\succsim, y, s_{-k}), y, s_{-k}).
\end{aligned}$$

Therefore, $y \notin \mathrm{DS}_k(\succsim, f')$, and hence $\mathrm{DS}_k(\succsim, f') = \{x\} = F_f(\succsim)$. Then, $\mathrm{DE}(\succsim, f') = \{(\underbrace{x, \ldots, x}_{m})\}$. Thus,

$$f'(\succsim, \mathrm{DE}(\succsim, f')) = \{x\} = F_f(\succsim).$$

$\square$

Proof of Theorem 3

Suppose that a domain $\mathscr{D} \subset \mathscr{R}^n$ admits a f-Condorcet winner. Take any $\succsim \in \mathscr{D}$. Let us show that for any $s \in \mathrm{NE}(\succsim, f)$, $s_1 = s_2 = \cdots = s_m$. Take any $s \in \mathrm{NE}(\succsim, f)$. Suppose that there exists $a_k, a_\ell \in A$ with $a_k \neq a_\ell$ such that $s_k \neq s_\ell$. Then, $f(\succsim, s) \neq s_k$ or $f(\succsim, s) \neq s_\ell$. Without loss of generality, suppose that $f(\succsim, s) \neq s_k$. Let $y \equiv f(\succsim, s_\ell, s_{-k})$. Then, by definition (i) of voting rule and $a_k \neq a_\ell$,

$$y \in \{s_1, \ldots, s_{k-1}, s_\ell, s_{k+1}, \ldots, s_m\} = \{s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_m\}.$$

Hence

$$\begin{aligned}\{s_1, \ldots, s_{k-1}, y, s_{k+1}, \ldots, s_m\} &= \{s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_m\} \\ &= \{s_1, \ldots, s_{k-1}, s_\ell, s_{k+1}, \ldots, s_m\}.\end{aligned}$$

Therefore, by definition (ii) of voting rule, $f(\succsim, y, s_{-k}) = f(\succsim, s_\ell, s_{-k}) = y$. Then, since $y = f(\succsim, y, s_{-k})$ and $s_k \neq f(\succsim, s)$, by equation (1),

$$\begin{aligned}u_k(f(\succsim, y, s_{-k}), y, s_{-k}) &\geq v_k(f(\succsim, y, s_{-k}), y, s_{-k}) + \frac{\bar{r}}{m} \\ &> v_k(f(\succsim, s_k, s_{-k}), s_k, s_{-k}) \\ &= u_k(f(\succsim, s_k, s_{-k}), s_k, s_{-k}),\end{aligned}$$

a contradiction to $s \in \mathrm{NE}(\succsim, f)$. Hence $s_1 = s_2 = \cdots = s_m$ for any $s \in \mathrm{NE}(\succsim, f)$.

Now, let $x \in X$ be such that $F_f(\succsim) = \{x\}$. We shall show that for any $s \in \mathrm{NE}(\succsim, f)$ and any $a_k \in A$, $s_k = x$. Suppose, by contradiction that, there exists $s \in \mathrm{NE}(\succsim, f)$ and $a_k \in A$ such that $s_k = y \neq x$. Then, for any $a_\ell \in A$, $s_\ell = y \neq x$.

Since x is a f-Condorcet winner at $\succsim$, $f(\succsim, x, s_{-k}) = x$. Moreover, $c(x, x, s_{-k}) = 1$. Then, by equation (1),

$$\begin{aligned}u_k(f(\succsim, x, s_{-k}), x, s_{-k}) &= v_k(f(\succsim, x, s_{-k})) + r_k(f(\succsim, x, s_{-k}), x, s_{-k}) \\ &= v_k(x) + \bar{r} \\ &\geq v_k(x) + \frac{\bar{r}}{m} + \frac{\bar{r}}{m} \\ &> v_k(f(\succsim, s_k, s_{-k})) + \frac{\bar{r}}{m} \\ &= v_k(f(\succsim, s_k, s_{-k})) + r_k(f(\succsim, s_k, s_{-k}), s_k, s_{-k}) \\ &= u_k(f(\succsim, s_k, s_{-k}), s_k, s_{-k}),\end{aligned}$$

a contradiction to $s \in \mathrm{NE}(\succsim, f)$. Hence for any $s \in \mathrm{NE}(\succsim, f)$ and any $a_k \in A$, $s_k = x$. Moreover, clearly $\mathrm{NE}(\succsim, f) = \{(x, \ldots, x)\}$. Then,

$$f(\succsim, \mathrm{NE}(\succsim, f)) = \{x\} = F_f(\succsim).$$

$\square$

Proof of Theorem 4

By definition of R_m, for any $L \in \Lambda^{I\times X}$ and any $s \in X^m$, there exists unique $x \in \{s_1, s_2, \ldots, s_m\}$ such that for any $y \in \{s_1, s_2, \ldots, s_m\}$, $xR_m(L)y$. Define a voting rule $f_m \in \mathscr{F}$ to satisfy that for any $L \in \Lambda^{I\times X}$, any $s \in X^m$ and any $y \in \{s_1, s_2, \ldots, s_m\}$, $f_m(L, s)R_m(L)y$. Let us show that for any $L \in \Lambda^{I\times X}$, $f_m(L, \mathrm{DE}(L, f_m)) = F_m(L)$. The proof parallels to that of Theorem 2.

Take any $L \in \Lambda^{I\times X}$. We shall show that for any $a_k \in A$, $\mathrm{DS}_k(L, f_m) = F_m(L)$. Take any $a_k \in A$. At first, note that there exists $x \in X$ such that $F_m(L) = \{x\}$. We first show that $x \in \mathrm{DS}_k(L, f_m)$. Take any $s_{-k} \in X^{m-1}$ and $s'_k \in X$. By definition of F_m, $x\, R_m(L)\, y$ for any $y \in \{s_1, \ldots, s_{k-1}, x, s_{k+1}, s_m\}$. Hence $f_m(L, x, s_{-k}) = x$. Suppose that $c(f_m(L, x, s_{-k}), x, s_{-k}) = 1$. Since for any $x, y \in X$, $v_k(x) = v_k(y)$,

$$\begin{aligned}
u_k(f_m(L, x, s_{-k}), x, s_{-k}) &= v_k(f_m(L, x, s_{-k})) + r_k(f_m(L, x, s_{-k}), x, s_{-k}) \\
&= v_k(x) + \bar{r} \\
&\geq v_k(x) + r_k(f_m(L, s'_k, s_{-k}), s'_k, s_{-k}) \\
&= v_k(f_m(L, s'_k, s_{-k})) + r_k(f_m(L, s'_k, s_{-k}), s'_k, s_{-k}) \\
&= u_k(f_m(L, s'_k, s_{-k}), s'_k, s_{-k}).
\end{aligned}$$

Next, suppose that $c(f_m(L, x, s_{-k}), x, s_{-k}) > 1$. If $s'_k = x$, then clearly

$$u_k(f_m(L, x, s_{-k}), x, s_{-k}) = u_k(f_m(L, s'_k, s_{-k}), s'_k, s_{-k}).$$

Therefore, we suppose that $s'_k \neq x$. Since $c(f_m(L, x, s_{-k}), x, s_{-k}) = c(x, x, s_{-k}) > 1$, there exists $a_\ell \in A$ with $a_\ell \neq a_k$ such that $s_\ell = x$. By $F_m(L) = \{x\}$ and $x \in \{s_1, \ldots, s_{k-1}, s'_k, s_{k+1}, \ldots, s_m\}$, $f_m(L, s'_k, s_{-k}) = x$. Then, since $x = f_m(L, x, s_{-k})$ and $s'_k \neq x = f_m(L, s'_k, s_{-k})$,

$$\begin{aligned}
u_k(f_m(L, x, s_{-k}), x, s_{-k}) &= v_k(f_m(L, x, s_{-k})) + r_k(f_m(L, x, s_{-k}), x, s_{-k}) \\
&= v_k(x) + \frac{\bar{r}}{c(x, x, s_{-k})} \\
&> v_k(x) + 0 \\
&= v_k(f_m(L, s'_k, s_{-k})) + r_k(f_m(L, s'_k, s_{-k}), s'_k, s_{-k}) \\
&= u_k(f_m(L, s'_k, s_{-k}), s'_k, s_{-k}).
\end{aligned}$$

Thus, $x \in \mathrm{DS}_k(L, f_m)$.

We next show that for any $y \in X$ with $y \neq x$, $y \notin \mathrm{DS}_k(L, f_m)$. Take any $y \in X$ with $y \neq x$. Let $s_{-k} \in X^{m-1}$ be such that $s_{-k} = (\underbrace{x, \ldots, x}_{m-1})$. Then, $f_m(L, y, s_{-k}) = x \neq y$ and $f_m(L, x, s_{-k}) = x$. Hence

$$u_k(f_m(L, x, s_{-k}), x, s_{-k}) > u_k(f_m(L, y, s_{-k}), y, s_{-k}).$$

Thus, $y \notin \mathrm{DS}_k(L, f_m)$, and hence $\mathrm{DS}_k(L, f_m) = \{x\} = F_m(L)$. Therefore, $\mathrm{DE}(L, f_m) = \{(\underbrace{x, \ldots, x}_{m})\}$. Thus,

$$f_m(L, \mathrm{DE}(L, f_m)) = \{x\} = F_m(L).$$

$\square$

Proof of Theorem 5

We show that f_m, which is defined in the proof of Theorem 4, implements the median rule F_m at Nash equilibria. The proof parallels to that of Theorem 3.

Take any $L \in \Lambda^{I \times X}$. Let us show that for any $s \in \mathrm{NE}(L, f_m)$, $s_1 = s_2 = \cdots = s_m$. Take any $s \in \mathrm{NE}(L, f_m)$. Suppose that there exists $a_k, a_\ell \in A$ with $a_k \neq a_\ell$ such that $s_k \neq s_\ell$. Then, $f_m(L, s) \neq s_k$ or $f_m(L, s) \neq s_\ell$. Without loss of generality, suppose that $f_m(L, s) \neq s_k$. Let $y \equiv f_m(L, s_\ell, s_{-k})$. Then, by definition (i) of voting rule,

$$y \in \{s_1, \ldots, s_{k-1}, s_\ell, s_{k+1}, \ldots, s_m\} = \{s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_m\}.$$

Hence

$$\begin{aligned}\{s_1, \ldots, s_{k-1}, y, s_{k+1}, \ldots, s_m\} &= \{s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_m\} \\ &= \{s_1, \ldots, s_{k-1}, s_\ell, s_{k+1}, \ldots, s_m\}.\end{aligned}$$

Therefore, by definition (ii) of voting rule, $f_m(L, y, s_{-k}) = f_m(L, s_\ell, s_{-k}) = y$. Then, by $y = f_m(L, y, s_{-k})$, $s_k \neq f_m(L, s_k, s_{-k})$ and equation (1),

$$u_k(f_m(L, y, s_{-k}), y, s_{-k}) > u_k(f_m(L, s_k, s_{-k}), s_k, s_{-k}),$$

a contradiction to $s \in \mathrm{NE}(L, f_m)$. Hence $s_1 = s_2 = \cdots = s_m$ for any $s \in \mathrm{NE}(L, f_m)$.

Let $x \in X$ be such that $F_m(L) = \{x\}$. Let us show that for any $s \in \mathrm{NE}(L, f_m)$ and any $a_k \in A$, $s_k = x$. Suppose, by contradiction that, there exists $s \in \mathrm{NE}(L, f_m)$ and $a_k \in A$ such that $s_k \neq x$. Then, for any $a_\ell \in A$, $s_\ell = s_k \neq x$.

By $F_m(L) = \{x\}$ and definition of f_m, $f_m(L, x, s_{-k}) = x$. Moreover,

$$c(f_m(L, x, s_{-k}), x, s_{-k}) = 1$$

while

$$c(f_m(L, s_k, s_{-k}), s_k, s_{-k}) = m.$$

Then, by equation (1),

$$
\begin{aligned}
u_k(f_m(L,x,s_{-k}),x,s_{-k}) &= v_k(f_m(L,x,s_{-k})) + r_k(f_m(L,x,s_{-k}),x,s_{-k}) \\
&= v_k(x) + \bar{r} \\
&\geq v_k(x) + \frac{\bar{r}}{m} + \frac{\bar{r}}{m} \\
&> v_k(f_m(L,s_k,s_{-k})) + \frac{\bar{r}}{m} \\
&= v_k(f_m(L,s_k,s_{-k})) + r_k(f_m(L,s_k,s_{-k}),s_k,s_{-k}) \\
&= u_k(f_m(L,s_k,s_{-k}),s_k,s_{-k}),
\end{aligned}
$$

a contradiction to $s \in \mathrm{NE}(L, f_m)$. Hence for any $s \in \mathrm{NE}(L, f_m)$ and any $a_k \in A$, $s_k = x$. Moreover, clearly $\mathrm{NE}(L, f_m) = \{(x, \ldots, x)\}$. Then,

$$f_m(L, \mathrm{NE}(L, f_m)) = \{x\} = F_m(L).$$

□

Proof of Proposition 1

Suppose that there exists an agenda setter $a_k \in A_1$ such that for any $x, y \in X$, $v_k(x) = v_k(y)$. Let f_m be identical to that of Proof of Theorem 4. We shall show that for any $L \in \Lambda^{I \times X}$ and any $s_{-k} \in X^{m-1}$,

$$f_m(L, \mathrm{DS}_k(f_m, L), s_{-k}) = F_m(L).$$

Take any $L \in \Lambda^{I \times X}$ and any $s_{-k} \in X^{m-1}$. In Proof of Theorem 4, we showed that $\mathrm{DS}_k(f_m, L) = F_m(L)$. Let $x \in X$ be such that $\mathrm{DS}_k(f_m, L) = F_m(L) = \{x\}$. Then, by definition of f_m, $f_m(L, \mathrm{DS}_k(f_m, L), s_{-k}) = \{x\} = F_m(L)$. □

Proof of Proposition 2

Suppose $|A_1| \geq 2$. Let $\bar{r}' \in \mathbb{R}_{++}$ be such that for any $a_k \in A_1$ and any $x, y \in X$,

$$v_k(x) + \bar{r} > v_k(y) + \frac{\bar{r}}{|A_1|}.$$

Note that since $|A_1| \geq 2$, such $\bar{r}$ exists. Let us show that for any $L \in \Lambda^{I \times X}$,

$$f_m(L, \mathrm{NE}(f_m, L)) = F_m(L).$$

Take any $L \in \Lambda^{I \times X}$. Let $x \in X$ be such that $F_m(L) = \{x\}$. The similar argument to Proof of Theorem 5 shows that for any $s \in \mathrm{NE}(L, f_m)$ and any $a_k \in A_1$, $s_k = x$. Moreover, obviously $\mathrm{NE}(L, f_m) = \{(x, \ldots, x)\}$. Then, by definition of f_m, $f_m(L, \mathrm{NE}(f_m, L)) = \{x\} = F_m(L)$. □

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