A Dynamic Auction that improves the Ausubel Auction*

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Abstract

Lawrence M. Ausubel (2004) introduces a new ascending-bid auction rule for multiple homogeneous objects, called *the Ausubel auction*, which is a dynamic counterpart of the Vickrey auction. He claims that in the Ausubel auction with private values, sincere bidding by all bidders is an *ex post perfect equilibrium*, which is a tuple of strategies constituting ex post equilibria at *all nodes* of the dynamic auction game. However, we show that this claim does not hold in general. In our counterexample, there exists a node at which sincere bidding by all bidders is *not* an ex post equilibrium. Furthermore, we examine properties of the sincere bidding equilibrium. Finally, we modify the Ausubel auction so that sincere bidding by all bidders is an ex post perfect equilibrium.

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1 Introduction

In his seminal work, Lawrence M. Ausubel (2004) designs a new ascending-bid auction rule for multiple homogeneous objects, called *the Ausubel auction*. This auction yields the Vickrey outcome at the sincere bidding equilibrium with private values. His main result claims that sincere bidding by all bidders is an *ex post perfect equilibrium* that constitutes ex post equilibria at *all nodes* of the dynamic auction game.

In this study, we show that this claim does not hold by giving a counterexample. That is, sincere bidding by all bidders is *not* always an ex post perfect equilibrium. In our counterexample of a dynamic auction game, there exists a subgame such that some bidder has an incentive not to sincerely bid (Theorem 1).

Next, we show that for any subgame, if a bidder does *not* bid more quantity than her demand just before the subgame, then she has an incentive to bid sincerely. Therefore, for any subgame, if each and every bidder does not bid more quantity than her demand just before the subgame, then sincere bidding by all bidders is an ex post equilibrium in the subgame(Theorem 2).

Finally, we modify the rule of the Ausubel auction so that sincere bidding by all bidders is an ex post perfect equilibrium. We introduce a new tie-breaking way such that each bidder can select whether she accepts an excessive supply. In this modified auction, sincere bidding by all bidders is an ex post perfect equilibrium (Theorem 3).

This paper is organized as follows. In Section 2, we introduce definitions. In Section 3, we give a counterexample to Ausubel's claim. In Section 4, we examine equilibrium properties of the Ausubel auction and give a modification. In Section 5, we conclude our discussion. Some proofs are relegated to Appendix.

2 Definitions

Our definitions and notation almost follow Section II and III of Ausubel (2004); however, we generalize some definitions so as to investigate details of the dynamic auction games.

2.1 Bidders

We construct a model of an auction for multiple objects with *private values.* A seller puts M *homogeneous goods* for auction.[1] A finite set of *bidders* is $N = \{1, 2, \dots, n\}$ with $n \geq 2$. Each bidder $i \in N$ has a *consumption set* $X_i = [0, \lambda_i]$ with $0 < \lambda_i \leq M$ and a *valuation function* $U_i : X_i \to \mathbb{R}_+$. When a bidder $i \in N$ is assigned $x_i \in X_i$ and pays $y_i \in \mathbb{R}$, bidder i's utility is $U_i(x_i) - y_i$. For each $x_i \in X_i$, the value $U_i(x_i)$ can be calculated by the integral of a corresponding *marginal value function* $u_i : X_i \to \{0, 1, \dots, \overline{u}\}$, so that

$$U_i(x_i) = \int_0^{x_i} u_i(q)dq \quad \forall x_i \in X_i.$$

Furthermore, we assume that each u_i is a *weakly decreasing* function in X_i, and $u_i(x_i)$ is an integer in $\{0, 1, \dots, \overline{u}\}$ for all $x_i \in X_i$.

2.2 The auction rule

We revisit the rule of the Ausubel auction with discrete times $\{0, 1, \dots, T\}$ where $T < \overline{u}$. For each time $t \in \{0, 1, \dots, T\}$, we define the price $p^t = t$. All bidders is informed of the price at each time. An auction starts at $t = 0$, and it proceeds as follows.

$\boldsymbol{t = 0}$**:** Each bidder $i \in N$ simultaneously bids a quantity $x_i^0 \in X_i$. If $\sum_{i \in N} x_i^0 \leq M$, then the auction ends at $t = 0$ with the assignment $(x_i^*)_{i \in N}$ such that

$$x_i^* = x_i^0 \quad \forall i \in N.$$

Otherwise, for each bidder $i \in N$, let

$$C_i^0 = \max\left\{0, M - \sum_{j \neq i} x_j^0\right\}$$

be bidder i's *cumulative clinches* at $t = 0$, and the auction continues to $t = 1$.

$\boldsymbol{t = s < T}$**:** The auctioneer announces information of prior bids to each bidder. Each bidder $i \in N$ simultaneously bids a quantity $x_i^s \in X_i$ satisfying the

[1] In this paper, we analyze the model for divisible goods, because we can treat a tie-breaking easily. To investigate the model for discrete goods, we need some additional assumptions on expected utilities.

biding constraint

$$C_i^{s-1} \leq x_i^s \leq x_i^{s-1}.$$

If $\sum_{i \in N} x_i^s \leq M$, the auction ends at $t = s$ with an assignment $(x_i^*)_{i \in N}$ such that

$$\sum_{i \in N} x_i^* = M$$
$$x_i^s \leq x_i^* \leq x_i^{s-1} \quad \forall i \in N.$$

Otherwise, let $C_i^s = \max\left\{0, M - \sum_{j \neq i} x_j^s\right\}$ be bidder i's *cumulative clinches* at $t = s$, and the auction continues to $s + 1$.

$\boldsymbol{t = T}$**:** The auctioneer announces information of prior bids to each bidder. Each bidder $i \in N$ simultaneously bids a quantity $x_i^T \in X_i$ with $C_i^{T-1} \leq x_i^T \leq x_i^{T-1}$. In any case, the auction ends. If $\sum_{i \in N} x_i^T > M$, an assignment $(x_i^*)_{i \in N}$ is such that $\sum_{i \in N} x_i^* = M$ and

$$x_i^* \leq x_i^T \quad \forall i \in N.$$

Otherwise, similarly to the case that ends at $t = s < T$, an assignment $(x_i^*)_{i \in N}$ is such that $\sum_{i \in N} x_i^* = M$ and $x_i^T \leq x_i^* \leq x_i^{T-1}$ for each $i \in N$.

This auction process finishes in at most $T + 1$ steps. Let L be the *last time* of an auction game, that is, $\sum_{i \in N} x_i^L \leq M$ or $L = T$. For each bidder $i \in N$, define cumulative clinches of the last time by i's assignment, $C_i^L = x_i^*$. Then, by this process, we obtain a vector of cumulative clinches $\left\{(C_i^t)_{i \in N}\right\}_{t=0}^L$. We define the vector of *current clinches* $\left\{(c_i^t)_{i \in N}\right\}_{t=0}^L$ as follows: For each $i \in N$ and $t \geq 1$,

$$c_i^t = C_i^t - C_i^{t-1},$$

and $c_i^0 = C_i^0$.

Each bidder's payment is calculated as follows: For each $i \in N$, the *payment* is given by

$$y_i = \sum_{t=0}^{L} p^t c_i^t.$$

In our study, the following rationing rule for a tie-breaking plays an important role, which is introduced by Ausubel in Footnotes 17 and 18. We consider two cases (i) $\sum_{i\in N} x_i^L < M$ with $L \geq 1$ and (ii) $\sum_{i\in N} x_i^L > M$ with $L = T$. In each case, there exist more than one assignment vector $(x_i^*)_{i\in N}$ such that $\sum_{j\in N} x_i^* = M$.

(i) $\boldsymbol{\sum_{i\in N} x_i^L < M}$ with $\boldsymbol{L \geq 1}$: If $x_i^L < x_i^{L-1}$, then i's assignment is more than her last bid, $x_i^* > x_i^L$. If i's last bid $\hat{x}_i^L > x_i^L$ increases and the last bids x_{-i}^L of the other bidders do not change, then i's assignment $\hat{x}_i > x_i^*$ must increase.

(ii) $\boldsymbol{\sum_{i\in N} x_i^L > M}$ with $\boldsymbol{L = T}$: If i's last bid $\hat{x}_i^L > x_i^L$ increases, and the last bids x_{-i}^L of the other bidders do not change, then the assignment i's $\hat{x}_i > x_i^*$ must increase.

This is a formulation for divisible goods. By changing the assigned quantity x_i^* to the expected assigned quantity $E(x_i^*)$, we can also apply the formulation to discrete goods.

2.3 Histories

At each $t \in \{1, \ldots, T+1\}$, a *history* h^t is a vector of prior bids to t

$$h^t = (x_1^s, x_2^s, \ldots, x_n^s)_{s\leq t-1} \in \big(\times_{i\in N} X_i\big)^{\{0,1,\ldots,t-1\}}$$

such that for each $i \in N$ and each $s \leq t-1$,

$$C_i^{s-1} \leq x_i^s \leq x_i^{s-1}, \tag{1}$$

$$\sum_{j\in N} x_j^{t-2} > M. \tag{2}$$

Define the history of starting point $t = 0$ by *empty sequence*, $h^0 = \emptyset$. Let H^t be the set of histories at t. Then, the set of all histories is given by $H \equiv \bigcup_{t=0}^{T+1} H^t$.

We call a history $z^{t+1} = (x_1^s, \ldots, x_n^s)_{s\leq t} \in H^{t+1}$ *terminal* if $\sum_{i\in N} x_i^t \leq M$ or $t = T$; i.e., $t = L$. Let Z^t be the set of terminal histories at t, and $Z \equiv \bigcup_{t=1}^{T+1} Z^t$ be the set of all terminal histories. We can see a terminal history as a *result of the auction game*, because this represents all bids from beginning to end. Then, for each $z \in Z$, an assignment vector $(x_i^*)_{i\in N}$ and a payment vector $(y_i)_{i\in N}$ are determined by the auction rule.

2.4 Strategies

At each time $t \in \{1, 2, \dots, T\}$, the auctioneer observes a history $h^t \in H^t \setminus Z^t$. Then, the auctioneer announces some information h_i^t to each bidder $i \in N$. Ausubel introduces three important *informational rules*; "full bid information," "aggregate bid information," and "no bid information." In this paper, we analyze auctions with *full bid information* so that each bidder $i \in N$ can observe all prior bids $h_i^t = h^t$ at each time t.

With full bid information, the set of observable histories of each bidder is $H \setminus Z$. Then, a strategy of bidder i is a function $\sigma_i : H \setminus Z \to X_i$ such that for any $t \in \{0, 1, \dots, T\}$ and $h^t = (x_1^s, x_2^s, \dots, x_n^s)_{s \le t-1} \in H^t \setminus Z^t$,

$$C_i^{t-1} \le \sigma_i(h^t) \le x_i^{t-1}$$

where $C_i^{t-1} = \max\{0, M - \sum_{j \ne i} x_j^{t-1}\}$.

For each $i \in N$, let Σ_i be the set of bidder i's strategies. For any n-tuple of strategies $(\sigma_i)_{i \in N} \in \times_{i \in N} \Sigma_i$, we attain a terminal history z^{L+1} which represents a result of an auction game. We denote $\pi_i((\sigma_j)_{j \in N})$ the utility of some bidder i at an n-tuple of strategies $(\sigma_j)_{j \in N}$.

We define *sincere bidding*, which is the strategy reporting truthfully the minimum demand at any history unless the bidder breaks the bidding constraint.

Definition 1. Bidder i's *sincere demand* at price $p \in \mathbb{Z}_+$ is defined by

$$Q_i(p) = \min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}.$$

Bidder i's *sincere bidding* is the strategy σ_i^* such that for any $t \ge 1$ and $h^t \in H^t \setminus Z^t$,

$$\sigma_i^*(h^t) = \min\{x_i^{t-1}, \max\{Q_i(p^t), C_i^{t-1}\}\},$$

and $\sigma_i^*(h^0) = Q_i(p^0)$.

We note that for each $p \in \{0, 1, 2, \dots, \overline{u}\}$, the existence of $Q_i(p)$ is guaranteed in this model. Moreover, the following property is satisfied.

Lemma 1. *For each $i \in N$ and $p \in \{1, 2, \dots, \overline{u}\}$,*

$$Q_i(p-1) = \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}.$$

Proof. See Appendix. □

By Lemma 1, for each $i \in N$, if the bidder plays the auction game sincerely, then an assignment x_i^* satisfies

$$\min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\} \leq x_i^* \leq \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}.$$

That is, although some bidder may be assigned more quantity than her last bid, the quantity maximizes her utility at the last price.

2.5 Subgames

We define *subgames* of the Ausubel auction with full bid information. Consider any $h \in H \setminus Z$. The set of histories in the subgame that follows h is given by

$$H|_h = \{h' \in H : h' = (h, h'') \text{ for some sequence } h''\}.$$

Then, the set of terminal histories in the subgame that follows h is given by

$$Z|_h = Z \cap H|_h.$$

For each $z \in Z$ and each $z' \in Z|_h$, if $z = z'$, then let the result of z' in the subgame that follows h be identical to the result of z in the original game.

A strategy of bidder i in the subgame that follows h is a function $\sigma_i : H|_h \setminus Z|_h \to X_i$ such that for any $h' = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq t-1} \in H|_h \setminus Z|_h$,

$$C_i^{t-1} \leq \sigma_i(h') \leq x_i^{t-1}$$

where $C_i^{t-1} = \max\{0, M - \sum_{j \neq i} x_j^{t-1}\}$. For each strategy $\sigma_i \in \Sigma_i$ in the original game, we denote $\sigma_i|_h \in \Sigma_i|_h$ the strategy which is induced in the subgame that follows h, that is, for each $h' \in H|_h \setminus Z|_h$, $\sigma_i|_h(h') = \sigma_i(h')$.

Let $\Sigma_i|_h$ be the set of bidder i's strategies in the subgame that follows h. As with the original game, we denote $\pi_i((\sigma_j)_{j \in N})$ bidder i's utility at an n-tuple of strategies of the subgame $(\sigma_j)_{j \in N} \in \times_{j \in N} \Sigma_j|_h$.

2.6 Ex post equilibrium

In the auction game, we usually assume that each bidder $i \in N$ knows only her own type and a probability distribution of types of all bidders. Then, we often investigate a Bayesian Nash Equilibrium or a stronger equilibrium, *ex post equilibrium*.[2] A tuple of strategies is an ex post equilibrium if it is a Bayesian Nash equilibrium with incomplete information and also a Nash equilibrium with complete information.

Note that, with private values, we can see a marginal valuation function u_i as a types of bidder $i \in N$. We denote a strategy $\sigma_i^{u_i}$ which depends on u_i.

Definition 2. For each $h \in H \setminus Z$, an n-tuple of strategies $(\sigma_j^{u_j})_{j \in N}$ is an *ex post equilibrium* in the subgame that follows h if for any $i \in N$, any $(u_j)_{j \in N}$ and any $\sigma_i \in \Sigma_i|_h$,

$$\pi_i((\sigma_i^{u_i}), (\sigma_j^{u_j})_{j \neq i}) \geq \pi_i(\sigma_i, (\sigma_j^{u_j})_{j \neq i}).$$

We can verify that $(\sigma_j^{u_j})_{j \in N}$ is a Nash equilibrium with complete information, as well as a Bayesian Nash equilibrium with incomplete information.

3 Counterexample

Ausubel (2004) extends Selten's perfectness concept of the extensive form game to the dynamic auction game. In extensive form games, a famous equilibrium notion is *subgame perfect equilibrium*, which is a tuple of strategies constituting Nash equilibria in all subgames. In contrast, we sometimes investigate an *ex post equilibrium* in auction games. Ausubel (2004) combines these two concepts and introduces the notion of *ex post perfect equilibrium*. Then, he claims that sincere bidding by all bidder is an ex post perfect equilibrium in the Ausubel auction.

Definition 3. An n-tuple of strategies $(\sigma_i)_{i \in N} \in \times_{i \in N} \Sigma_i$ is an *ex post perfect equilibrium* if for each $h \in H \setminus Z$, $(\sigma_i|_h)_{i \in N}$ is an ex post equilibrium in the subgame that follows h.

[2] See Crémer, Jacques and McLean. Krishna (2009) provides an explanation for the notion of ex post equilibrium. He summarizes a relationship between ex post equilibria, Bayesian Nash equilibria and weakly dominant strategy equilibria.

Claim 1 (Ausubel 2004, Theorem 1)**.** *In the alternative ascending-bid auction with private values, sincere bidding by all bidders is an ex post perfect equilibrium, yielding the efficient outcome of the Vickrey auction.*

However, this claim does not hold in general. We give a counterexample such that sincere bidding by all bidders is not an ex post perfect equilibrium.

Theorem 1. *In the Ausubel auction, sincere bidding by all bidders is not always an ex post perfect equilibrium.*

Proof. Consider the case with two bidders and six quantity of an object. Let u_1, u_2 be the marginal value functions of two bidders such that

$$u_1(q) = u_2(q) = \begin{cases} 6 & \text{if } q \in [0,1) \\ 1 & \text{if } q \in [1,6]. \end{cases}$$

Consider the history $h^5 = (x_1^t, x_2^t)_{t=0,1,2,3,4} = \big((6,6),(6,6),(6,6),(6,6),(6,6)\big)$.

Sincere bidding: z^6

After h^5, the bidders report $x_1^6 = x_2^6 = 1$ with sincere bidding. Then, we have $z^6 = (h^5, (1,1))$. Since $x_1^6 + x_2^6 = 2 < M = 6$, the auction ends at z^6, yielding an assignment (x_1^*, x_2^*) such that

$$\begin{aligned} 1 \leq x_1^* \leq 6, \\ 1 \leq x_2^* \leq 6, \\ x_1^* + x_2^* = 6. \end{aligned}$$

Thus, there exists a bidder whose quantity of this assignment is at least three. Without loss of generality, $x_1^* \geq 3$. Since bidder 1 did not clinch at $t \leq 4$, the payment of 1 is $y_1^* = 5 \cdot x_1^*$. Thus, 1's expected utility of z^6 is $U_i(x_1^*) - 5 \cdot x_1^*$.

Misreporting: $\hat{z}^6$

On the other hand, if bidder 1 reports $\hat{x}_1^5 = 0$ with misreporting, and bidder 2 reports $x_2^5 = 1$ after h^5, then the auction ends at $\hat{z}^6 = (h^5, (0,1))$, yielding an assignment $(\hat{x}_1, \hat{x}_2)$ such that

$$\begin{aligned} 0 \leq \hat{x}_1 \leq 6, \\ 1 \leq \hat{x}_2 \leq 6, \\ \hat{x}_1 + \hat{x}_2 = 6. \end{aligned}$$

By the monotone property of rationing rules, since $0 = \hat{x}_1^5 < x_1^5 = 1$ and any other condition of $\hat{z}^6$ is the same as z^6, the quantity of assignment $\hat{x}_1$ must be strictly less than x_1^*. Note that similarly to z^6, 1's utility of $\hat{z}^6$ is $U_i(\hat{x}_1) - 5 \cdot \hat{x}_1$.

We calculate the difference between 1's utilities of z^6 and $\hat{z}^6$,

$$\begin{aligned}
&[U_i(x_1^*) - 5 \cdot x_1^*] - [U_i(\hat{x}_1) - 5 \cdot \hat{x}_1] \\
&= \left[\int_0^{x_1^*} u_1(q)dq - 5 \cdot x_1^*\right] - \left[\int_0^{\hat{x}_1} u_1(q)dq - 5 \cdot \hat{x}_1\right] \\
&= \left[\int_0^{x_1^*} u_1(q)dq - \int_0^{\hat{x}_1} u_1(q)dq\right] - 5 \cdot \left[x_1^* - \hat{x}_1\right] \\
&= \int_{\hat{x}_1}^{x_1^*} u_1(q)dq - 5 \cdot \left[x_1^* - \hat{x}_1\right]. \qquad (3)
\end{aligned}$$

Case 1: $\hat{\boldsymbol{x}}_1 \geq 1$. We calculate (3) such that

$$x_1^* - \hat{x}_1 - 5 \cdot \left[x_1^* - \hat{x}_1\right] = -4 \cdot \left[x_1^* - \hat{x}_1\right] < 0.$$

Case 2: $\hat{\boldsymbol{x}}_1 \in (0, 1)$. We calculate (3) such that

$$\begin{aligned}
&\left[x_1^* - 1\right] + 6 \cdot \left[1 - \hat{x}_1\right] - 5 \cdot \left[x_1^* - \hat{x}_1\right] \\
&= -4 \cdot x_1^* - \hat{x}_1 + 6 < 0 \qquad (\because x_1^* \geq 3).
\end{aligned}$$

Thus, 1's utility of $\hat{z}^6$ is strictly greater than that of z^6. Therefore, sincere bidding by all bidder is not an ex-post perfect equilibrium. □

4 Amending the result

In Section 2, we give a counterexample to the result of Ausubel. In this section, we amend this result. First, we examine equilibrium properties. Next, we modify the rule of the Ausubel auction.

4.1 Examining equilibrium properties

In our counterexample, we consider a history such that some bidder bids more quantity than her demand just before the history. Then, an assignment of the bidder may be more than $Q_i(p^{L-1}) = \sup\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\}$, even if her last bid is $Q_i(p^L)$. Thus, the inequality

$$\min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\} \leq x_i^* \leq \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}.$$

does not hold. Therefore, in this case, there may be an incentive *not* to sincerely bid in the subgame that follows the history.

However, if for a history, a bidder does not bid more quantity than her sincere demand just before the history, the bidder has an incentive to sincerely bid in the subgame that follows the history.

Lemma 2. *Given any $(u_j)_{j \in N}$, for each $j \in N$, let σ_j^* be sincere bidding corresponding to u_j. Consider any $t \in \{0, 1, \dots, T\}$, any $h^t = (x_1^s, x_2^s, \dots, x_n^s)_{s \leq t-1} \in H^t \setminus Z^t$, and any $i \in N$. If $x_i^{t-1} \leq Q_i(p^{t-1})$, then for any $\sigma_i \in \Sigma_i|_{h^t}$,*

$$\pi_i(\sigma_i^*|_{h^t}, \sigma_{-i}^*|_{h^t}) \geq \pi_i(\sigma_i, \sigma_{-i}^*|_{h^t}).$$

Proof. See Appendix. □

Applying this Lemma to all bidders, we have the following result.

Theorem 2. *If a history $h^t = (x_1^s, x_2^s, \dots, x_n^s)_{s \leq t-1} \in H^t \setminus Z^t$ is such that for each $i \in N$, $x_i^{t-1} \leq Q_i(p^{t-1})$, sincere bidding by all bidders is an ex post equilibrium in the subgame that follows h.*

Proof. Immediately follows from Lemma 2. □

4.2 Modifying the Auction rule

In the original Ausubel auction, bidders may be assigned more quantity than their last bid without being asked whether they want it. In *the modified Ausubel auction*, at each time, each bidder reports quantity and selects whether she accepts an excessive supply. If at the last time of the auction, a bidder reports that she does not want to be assigned more quantity than her last bid, then she is assigned only her last bid.

We use the same model in Section 2. The process of the modified Ausubel auction is as follows.

$\boldsymbol{t = 0}$**:** Each bidder $i \in N$ simultaneously reports quantity $x_i^0 \in X_i$ and a *signal* $a_i^0 \in \{0, 1\}$ for *tie-breaking*. If $\sum_{i \in N} x_i^0 \leq M$, the auction ends at $t = 0$

with the assignment $(x_i^*)_{i \in N}$ which is

$$x_i^* = x_i^0 \quad \forall i \in N.$$

Otherwise, for each bidder $i \in N$, let

$$C_i^0 = \max\left\{0, M - \sum_{j \neq i} x_j^0\right\}$$

be bidder i's cumulative clinches at $t = 0$, and the auction continues to $t = 1$.

***t* = *s* < *T*:** The auctioneer announces information of prior bids to each bidder. Each bidder $i \in N$ simultaneously reports quantity $x_i^s \in X_i$ satisfying the constraint

$$C_i^{s-1} \leq x_i^s \leq x_i^{s-1},$$

and a signal $a_i^s \in \{0, 1\}$. If $\sum_{i \in N} x_i^s \leq M$, the auction ends at $t = s$ with an assignment $(x_i^*)_{i \in N}$ which is decided in the following way: Let $N_0 = \{i \in N : a_i = 0\}$ and $N_1 = \{i \in N : a_i = 1\}$. Then

$$x_i^* = x_i^s \qquad \forall i \in N_0$$
$$x_i^* = x_i^s + \min\{x_i^{s-1} - x_i^s, \frac{x_i^{s-1} - x_i^s}{\sum_{k \in N_1} x_k^{s-1} - x_k^s} \cdot \left(M - \sum_{i \in N} x_i^s\right)\} \qquad \forall i \in N_1$$

Otherwise, let $C_i^s = \max\left\{0, M - \sum_{j \neq i} x_j^s\right\}$ be bidder i's cumulative clinches, and the auction continues to $s + 1$.

***t* = *T*:** The auctioneer announces information of prior bids to each bidder. Each bidder $i \in N$ simultaneously bids quantity $x_i^T \in X_i$ with $C_i^{T-1} \leq x_i^T \leq x_i^{T-1}$ and signal $a_i^T \in \{0, 1\}$. In any case, the auction ends, even when there is excess demand. If $\sum_{i \in N} x_i^T > M$, an assignment $(x_i^*)_{i \in N}$ is such that $\sum_{i \in N} x_i^* = M$ and

$$x_i^* \leq x_i^T \quad \forall i \in N.$$

Otherwise, as with to the case that ends at $t = s < T$ an assignment $(x_i^*)_{i \in N}$ is decided.

Note that M homogeneous good may *not* be assigned *entirely* by the modified Ausubel auction. However, we can assign M homogeneous good entirely and achieve an efficient outcome at sincere bidding equilibrium by defining sincere bidding as in Definition 4. Moreover, in the modified Ausubel auction, sincere bidding by all bidders is an ex post perfect equilibrium.

Definition 4. Bidder i's *sincere bidding in the modified Ausubel auction* is the strategy σ_i^* such that for any $t \geq 1$ and $h^t \in H^t \setminus Z^t$,

$$\sigma_i^*(h^t) = (\min\{x_i^{t-1}, \max\{Q_i(p^t), C_i^{t-1}\}\}, 1_{x_i^{t-1} \leq Q_i(p^{t-1})}),$$

and $\sigma_i^*(h^0) = (Q_i(p^0), 1)$.

Theorem 3. *In the modified Ausubel auction, sincere bidding by all bidders is an ex post perfect equilibrium.*

Proof. See Appendix. □

5 Conclusion

In this paper, we have investigated sincere bidding equilibrium in the Ausubel auction. We first gave a counterexample to the main result of Ausubel (2004). That is, we showed that sincere bidding by all bidders is not always an ex post perfect equilibrium. We then amended this result. In other words, we showed if a bidder does not bid her demand just before a node, then she has an incentive to sincerely bid at the node. Then, if each and every bidder does not bid her demand just before a node, sincere bidding by all bidders is an ex post equilibrium in the node. We also modified the Ausubel auction so that sincere bidding by all bidders is an ex post perfect equilibrium.

Appendix: Proofs

Proof of lemma 1. Since u_i is a weakly decreasing integer-valued function, there is a partition $\{a_0, \dots, a_m\} \subset X_i$ with $0 = a_0 < \cdots < a_m = \lambda_i$ and values $\{b_1, \dots, b_m\} \subset \{0, 1, \dots, \overline{u}\}$ with $b_1 > b_2 > \cdots > b_m$ such that for each k with $1 \leq k \leq m$,

$$u_i(x_i) = b_k \text{ if } a_{k-1} < x_i < a_k.$$

Note that $m \leq T$. Consider any $x_i' \in X_i$. Let

$$k = \arg\min_{\ell}\{a_\ell : x_i' \geq a_\ell\}.$$

Then, by the definition of Riemann Integral,

$$U_i(x_i') = \int_0^{x_i'} u_i(q)dq = \sum_{\ell=1}^{k-1} b_\ell(a_\ell - a_{\ell-1}) + b_k(x_i - a_{k-1}). \tag{4}$$

Take any $p \in \{1, \dots, T\}$. Define $b_0 = T + 1$. Let

$$r = \arg\min_{\ell}\{b_\ell : p < b_\ell\},$$
$$r' = \arg\min_{\ell}\{b_\ell : p \leq b_\ell\}.$$

By equation (4), we can verify that

$$a_r = \min\{\arg\max_{x_i \in X_i} U_i(x_i) - px_i\},$$
$$a_{r'} = \max\{\arg\max_{x_i \in X_i} U_i(x_i) - px_i\}.$$

Because $b_\ell \in \mathbb{Z}$ for each ℓ,

$$\{b_\ell : p - 1 < b_\ell\} = \{b_\ell : p \leq b_\ell\}.$$

Therefore,

$$\min\{\arg\max_{x_i \in X_i} U_i(x_i) - (p-1)x_i\} = \max\{\arg\max_{x_i \in X_i} U_i(x_i) - px_i\}.$$

□

Proof of Theorem 2. Consider any $t \in \{0, 1, \dots, T\}$,

$$h^t = \{(x_1^s, x_2^s, \dots, x_n^s)\}_{s \le t-1} \in H^t \setminus Z^t,$$

and $(u_j)_{j \in N}$. For each $j \in N$, let σ_j^* be sincere bidding which is corresponding to u_j, and $\sigma_j^*|_{h^t}$ be induced sincere bidding in the subgame that follows h^t.

Take any $i \in N$ and $\sigma_i \in \Sigma_i|_{h^t}$. Suppose that $x_i^{t-1} \le Q_i(p^{t-1})$. We shall show that

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \ge \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \ne i}).$$

Let

$$z^{L+1} = (x_1^s, x_2^s, \dots, x_n^s)_{s \le L}$$

be the terminal history which is reached by $(\sigma_j^*|_{h^t})_{j \in N}$, and

$$w^{L'+1} = (\hat{x}_1^s, \hat{x}_2^s, \dots, \hat{x}_n^s)_{s \le L'}$$

be the terminal history which is reached by $(\sigma_i, (\sigma_j^*|_{h^t})_{j \ne i})$. Denote $\{(C_j^t)_{j \in N}\}_{t=0}^{L}$ the cumulative clinches of z^{L+1}, and $\{(\hat{C}_j^t)_{j \in N}\}_{t=0}^{L'}$ the cumulative clinches of $w^{L'+1}$.

Step 1. For each $s \in \{t-1, \dots, L-1\}$, $x_i^s \le Q_i(p^s)$.

If $s = t-1$, $x_i^{s-1} = x_i^{t-1} \le Q_i(p^{t-1}) = Q_i(p^{s-1})$. Then, we consider the case $s \ge t$. By the definition of sincere bidding,

$$x_i^s = \sigma_i^*|_{h^t}((x_1^\ell, \dots, x_n^\ell)_{\ell \le s-1}) = \min\{x_i^{s-1}, \max\{C_i^{s-1}, Q_i(p^s)\}\}.$$

Suppose that $x_i^s = C_i^{s-1} > 0$. Then, $x_i^{s-1} = M - \sum_{j \ne i} x_j^{s-2}$. Because $x_j^s \le x_j^{s-1}$ for each $j \in N$, $\sum_{j \in N} x_j^s < M$. Then, the auction ends at s in the history z^{L+1}. This contradicts that the auction ends at $s \le L-1$. Thus, $x_i^s \ne C_i^{s-1} > 0$. Then, we have $x_i^s = \min\{x_j^{s-1}, Q_j(s)\}$. Therefore, $x_i^s \le Q_i(p^s)$.

Step 2. For each $j \ne i$ and $s \in \{t, \dots, \min\{L-1, L'-1\}\}$,

$$x_j^s = \hat{x}_j^s.$$

We shall prove step 2 by induction. Consider any $j \neq i$.
At $s = t$:Because $x_j^t = \sigma_j^*|_{h^t}(h^t)$ and $\hat{x}_j^t = \sigma_j^*|_{h^t}(h^t)$, obviously $x_j^t = \hat{x}_j^t$.

At $s = k$ $(t+1 \leq k \leq \min\{L-1, L'-1\})$: Suppose that $x_j^\ell = \hat{x}_j^\ell$ for all ℓ with $t+1 \leq \ell \leq k-1$. By the definition of sincere bidding,

$$x_j^k = \sigma_j^*|_{h^t}((x_1^\ell, \ldots, x_n^\ell)_{\ell \leq k-1}) = \min\{x_j^{k-1}, \max\{C_j^{k-1}, Q_j(p^k)\}\},$$
$$\hat{x}_j^k = \sigma_j^*|_{h^t}((\hat{x}_1^\ell, \ldots, \hat{x}_n^\ell)_{\ell \leq k-1}) = \min\{\hat{x}_j^{k-1}, \max\{\hat{C}_j^{k-1}, Q_j(p^k)\}\}.$$

Note that $x_j^k \neq C_j^{k-1} > 0$ and $\hat{x}_j^k \neq \hat{C}_j^{k-1} > 0$.[3] Thus, $x_j^k = \min\{x_j^{k-1}, Q_j(k)\}$ and $\hat{x}_j^k = \min\{\hat{x}_j^{k-1}, Q_j(k)\}$. By the assumption of induction $x_j^{k-1} = \hat{x}_j^{k-1}$, we have $x_j^k = \hat{x}_j^k$.

Next we consider the following there cases; $L = L'$, $L > L'$ and $L < L'$.

Step 3: $L = L'$. By step 2, we have for all $s \leq L-1 = L'-1$ and $j \neq i$, $x_j^s = \hat{x}_j^s$. Hence, for all $s \leq L-1$, $C_i^s = \hat{C}_i^s$. We then calculate C_i^L and $\hat{C}_i^{L'}$.

Step 3-1: $Q_i(p^L) \leq x_i^L$. Then, $x_i^L = C_i^{L-1}$. Since the auction does not end at $L-1$ in the history z^{L+1}, by step 1, $x_i^{L-1} \leq Q_i^{L-1}$. Thus,

$$Q_i(p^L) \leq x_i^L \leq C_i^L \leq x_i^{L-1} \leq Q_i(p^{L-1}).$$

Therefore,

$$\min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\} \leq C_i^L \leq \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}.$$

Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Step 3-2: $Q_i(p^L) > x_i^L$. We shall show that $x_i^L = x_i^{t-1}$. By the definition of sincere bidding,

$$x_i^L = \sigma_i^*|_{h^t}((x_1^s, \ldots, x_n^s)_{s \leq L-1}) = \min\{x_i^{L-1}, \max\{C_i^{L-1}, Q_i(p^L)\}\}.$$

Since $x_i^L < Q_i(p^L)$, $x_i^L = x_i^{L-1}$. Because $Q_i(p^L) \leq Q_i(p^{L-1})$, $x_i^{L-1} < Q_i(p^{L-1})$. If $L-1 = t-1$, $x_i^{L-1} = x_i^{t-1}$. On the other hand, by the definition of sincere bidding, we have $x_i^{L-1} = x_i^{L-2}$. By repeating this procedure, $x_i^L = x_i^{L-1} = \cdots = x_i^{t-1}$.

[3] One can easily check that if some bidder bids a *positive* cumulative clinches, the auction imediately ends at the time. Then, it contradict to $k \leq \min\{L-1, L'-1\}$.

Since bidder i cannot bid more quantity than x_i^{t-1} after $t-1$, $\hat{C}_i^L \leq x_i^{t-1}$. Then,

$$\hat{C}_i^{L-1} \leq C_i^{L-1} < \min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}$$

Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

Step 4: $\boldsymbol{L > L'}$. We remark that for each $s \leq L'-1$, $C_i^s = \hat{C}_i^s$. Then, we calculate $\{C_i^s\}_{s=L'}^L$ and $\hat{C}_i^{L'}$. Since the auction does not end at L' in the history z^{L+1}, each bidder $j \neq i$ does not bid a positive cumulative cliches $C_j^{L'-1}$ at L'. Then,

$$x_j^{L'} = \min\{x_j^{L'-1}, Q_j(p^{L'})\}.$$

On the other hand,

$$\hat{x}_j^{L'} = \min\{\hat{x}_j^{L'-1}, \max\{\hat{C}_j^{L'-1}, Q_j(p^{L'})\}\}.$$

Since $x_j^{L'-1} = \hat{x}_j^{L'-1}$, $x_j^{L'} \leq \hat{x}_j^{L'}$. Thus, $\hat{C}_i^{L'} \leq C_i^{L'}$. Note that by step 1, $C_i^{s'} \leq Q_i(p^{s'})$ for all $s \in \{L', \dots, L-1\}$, and $Q_i(p^L) \leq C_i^L \leq Q_i(p^{L-1})$. Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

Step 5: $\boldsymbol{L < L'}$. We first show that $Q_i(p^L) \leq x_i^L$. Suppose that $Q_i(p^L) > x_i^L$. Similarly to step 3-2 $x_i^L = x_i^t$. Then, $\hat{x}_i^L \leq x_i^{L'}$, and the auction ends at L in the history $w^{L'+1}$. This contradicts to $L' > L$. Thus, $Q_i(p^L) \leq x_i^L$.

Next, we remark that for each $s \leq L-1$, $C_i^s = \hat{C}_i^s$. Similarly to step 4, we have $C_i^L \leq \hat{C}_i^L$. Then, $Q_i(p^L) \leq C_i^L \leq \hat{C}_i^L$. Moreover, for each $s \geq L$ because $\hat{C}_i^L \leq C_i^s$ and $Q_i(p^s) \leq Q_i(p^L)$, $Q_i(p^s) \leq C_i^s$. Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

□

Proof of Theorem 3. Consider any $t \in \{0, 1, \dots, T\}$,

$$h^t = \big((x_1^s, a_1^s), (x_2^s, a_2^s) \dots, (x_n^s, a_n^s)\big)_{s \leq t-1} \in \left(\times_{i\in N} \big(X_i, \times\{0,1\}\big)\right)^{\{0,1,\dots,t-1\}}$$

and $(u_j)_{j\in N}$. For each $j \in N$, let σ_j^* be sincere bidding which is corresponding to u_j, and $\sigma_j^*|_{h^t}$ be induced sincere bidding in the subgame that follows h^t.

Take any $i \in N$ and $\sigma_i \in \Sigma_i|_{h^t}$. If $x_i^{t-1} \leq Q_i(p^{t-1})$, then we can show similarly to Proposition 2. Suppose that $x_i^{t-1} > Q_i(p^{t-1})$.

Let

$$z^{L+1} = \Big((x_1^s, a_1^s)(x_2^s, a_2^s), \ldots, (x_n^s, a_n^s)\Big)_{s\leq L}$$

be the terminal history which is reached by $(\sigma_j^*|_{h^t})_{j\in N}$, and

$$w^{L'+1} = \Big((\hat{x}_1^s, \hat{a}_1^s)(\hat{x}_2^s, \hat{a}_2^s), \ldots, (\hat{x}_n^s, \hat{a}_n^s)\Big)_{s\leq L'}$$

be the terminal history which is reached by $(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i})$. Denote $\{(C_j^t)_{j\in N}\}_{t=0}^{L}$ the cumulative clinches of z^{L+1}, and $\{(\hat{C}_j^t)_{j\in N}\}_{t=0}^{L'}$ the cumulative clinches of $w^{L'+1}$.

We first consider the case $L > t$. Since the auction does not end at t in the history z^{L+1},

$$x_i^t \neq C_i^{t-1}.$$

Because $Q_i(p^t) \leq Q_i(p^{t-1}) < x_i^{t-1}$,

$$x_i^t \neq x_i^{t-1}.$$

That is, $x_i^t = Q_i(p^t)$. If the auction ends at t in the history $w^{L'+1}$,

$$\hat{C}_i^{L'+1} \leq C_i^t \leq Q_i(p^t) = \min\{\arg\max_{x_i\in X_i}(U_i(x_i) - px_i)\}.$$

On the other hand, the equation $\hat{C}_i^{t+1} = C_i^t$ holds. Then, since $x_i^t = Q_i(p^t)$, we can prove the case $L < t$ similarly to Proposition 1.

We next consider the case $L = t$. Then, $\sigma_i^*|_{h^t}(h^t) = (\max(C_i^{t-1}, Q_i(p^t)), 0)$, because $x_i^{t-1} > Q_i(p^{t-1}) \geq Q_i(p^t)$. Thus,

$$C_i^L = \max\{C_i^{t-1}, Q_i(p^t)\}.$$

Then

$$\hat{C}_i^L \geq C_i^L \geq Q_i(p^t) = \min\{\arg\max_{x_i\in X_i}(U_i(x_i) - px_i)\}.$$

Then for all $s \leq t$, the clinches $\hat{C}_i^s \geq \min\{\arg\max_{x_i \in X_i}(U_i(x_i) - px_i)\}$ $s \leq t$ reduce the utility of bidder i. Thus,

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

□

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