Core and competitive equilibria: an approach from discrete convex analysis

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Abstract

We extend the assignment market (Shapley and Shubik (1972), Kaneko (1976, 1982)) by utilizing discrete convex analysis. We consider the market in which buyers and sellers trade indivisible commodities for money. Each buyer demands at most one unit of commodity. Each seller produces multiple units of several types of commodities. We assume that the cost function of each seller is *$M^{\natural}$-convex*, which is a concept in discrete convex analysis. We make the quasi-linearity assumption on the sellers, but not on the buyers. We prove that the Core and and the competitive equilibria exist and coincide in our market model.

JEL classification: C71, D41
Keywords: competitive equilibrium; core; discrete convex analysis; $\mathrm{M}^{\natural}$-convex function

1 Introduction

The assignment market (Shapley and Shubik (1972), Kaneko (1976, 1982)) is a prominent model for the study of markets in which buyers and sellers trade indivisible commodities for money. The trade is bilateral, and the demand of buyers is binary, i.e., buyers demand at most one unit of commodity. The main result of the assignment market is that the Core and the competitive equilibria coincide. Compared with the case of continuous commodity space, we can guarantee the coincidence without assuming infinite

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'copies' of agents. The result gives us a clear insight into the structure of the competitive equilibria.

We refer to previous works. Shapley and Shubik (1972) formulated the market and proved the existence and the coincidence between the two concepts by using the duality theorem of the linear programming problem. Kaneko (1976) extended the model to cases where sellers can produce multiple units of several types of commodities. It is assumed that the cost function of each seller is *separable convex*, i.e., the cost function is represented as the sum of univariate functions with non-decreasing marginal costs. In the above two papers, the quasi-linearity assumption is imposed on utility functions of buyers. This assumption implies no income effect, which is not suitable for the market where the proportion of price to income is not negligible. Kaneko (1982) extended previous models to cases without the quasi-linearity assumption. Kaneko's (1982) model is called the generalized assignment market, abbreviated as the GAM model hereafter.

The purpose of this paper is to further extend the GAM model. In the GAM model, sellers have separable convex cost functions. However, this assumption is restrictive in the following sense: if a seller has a separable convex function, then the cost of producing one commodity is always independent of the production of other types of commodities. In order to relax this assumption, we describe convexity assumption in a different way. We assume that the cost function of each seller is *$M^\natural$-convex*, which is a concept in discrete convex analysis (Murota (2003)). $\mathrm{M}^\natural$-convex function contains separable convex function as a special case, and allows the cost of a commodity to be dependent on the production of other types of commodities.

Discrete convex analysis is a branch of discrete mathematics, and studies several types of convex functions on discrete domains. $\mathrm{M}^\natural$-convex function is one of the major concepts in the theory. It has been applied to market theory or matching theory in the literature. Murota (2003) proved the existence of a competitive equilibrium in the market where agents trade indivisible commodities for money and have $\mathrm{M}^\natural$-concave utility functions. Fujishige and Tamura (2007) gave a generalization of Shapley and Shubik's (1972) assignment market and Gale and Shapley's (1962) marriage market by utilizing discrete convex analysis. Kojima et al. (2015) applied discrete convex analysis to two-sided matching market in which certain distributional constraints exist. They proved that, if the preferences of hospitals can be represented by an $\mathrm{M}^\natural$-concave function, then the generalized deferred acceptance mechanism is strategyproof and yields a stable matching.

To our best knowledge, this paper is the first attempt to study the coincidence between the Core and the competitive equilibria in the context of $\mathrm{M}^\natural$-convexity. We explain our main result. Consider the market in which buyers demand at most one unit of commodity, and sellers have M$^\natural$-convex cost functions. In this market, under some conditions, the Core and the competitive equilibria exist and coincide.

The remaining part is organized as follows. In Section 2, we introduce our market model. In Section 3, we define M$^\natural$-convex function. We also refer to previous studies on the relationship between the gross-substitutes condition and M$^\natural$-concavity. Section 4 presents the main results. Section 5 gives concluding remarks. All proofs are in the Appendix.

2 Market model

2.1 Buyers and sellers

Let H denote the set of *buyers*, J denote the set of *sellers*, and L denote the set of *commodities.* The three sets are non-empty and finite.

For each $l \in L$, let $\mathbf{1}_l$ denote the l-th unit vector, and $\mathbf{1}_0$ denote the 0-vector. We define

$$X = \{\mathbf{1}_l : l \in L \cup \{0\}\}.$$

We define the *consumption set* by $X \times \mathbb{R}_+$. An element $(\mathbf{1}_l, c) \in X \times \mathbb{R}_+$ means that a buyer consumes one unit of commodity l and c amount of money. Let $I^h \geq 0$ denote the *income* of buyer $h \in H$. For each $h \in H$ and $p \in \mathbb{R}^L_+$, we define

$$X^h_p = \{x \in X : p \cdot x \leq I^h\}.$$

In words, X^h_p is the set of commodities that h can consume at price vector p.

A buyer $h \in H$ has a utility function $U^h : X \times \mathbb{R}_+ \to \mathbb{R}$. We make the following assumptions:

A1 (*Monotonicity and continuity*). $U^h(\cdot, \cdot)$ is strictly monotonic and continuous with respect to the second argument.

A2 (*Indispensability of money*). For each $x \in X$, $U^h(\mathbf{0}, I^h) \geq U^h(x, 0)$.

For each $h \in H$, we define the *demand correspondence* $D^h : \mathbb{R}^L_+ \rightrightarrows X$ by

$$D^h(p) = \arg\max_{x \in X^h_p} U^h(x, I^h - p \cdot x) \text{ for all } p \in \mathbb{R}^L_+.$$

A seller $j \in J$ has a cost function $C^j : \mathbb{Z}^L_+ \to \mathbb{R} \cup \{+\infty\}$. Here, $C^j(x) = +\infty$ means that j cannot produce x. Let $Y^j = \{x \in \mathbb{Z}^L_+ : C^j(x) < +\infty\}$. We assume that $C^j(\mathbf{0}) = 0$ and Y^j is bounded. We also assume the following:

$$\text{for each } l \in L, \text{ there exists } j \in J \text{ such that } \mathbf{1}_l \in Y^j. \tag{1}$$

Namely, for each commodity, there is at least one seller who can produce the commodity.

For two vectors $x, y \in \mathbb{Z}^L$, $x \geq y$ means that $x_l \geq y_l$ for all $l \in L$. We make the following assumption:

A3 (*Monotonicity*). $C^j(\cdot)$ is monotone-nondecreasing, i.e., for any $x, y \in Y^j$, $x \geq y$ implies $C^j(x) \geq C^j(y)$.

For each $j \in J$, we define the *supply correspondence* $S^j : \mathbb{R}^L_+ \rightrightarrows Y^j$ by

$$S^j(p) = \arg\max_{x \in Y^j} \{p \cdot x - C^j(x)\} \text{ for all } p \in \mathbb{R}^L_+.$$

In Section 3, we additionally make convexity assumption on C^j.

2.2 Remark: Sellers with initial endowments

In Subsection 2.1, we assumed that sellers have cost functions and produce commodities. We briefly explain that our market model subsumes the case in which sellers have utility functions over initial endowments.

Consider the market in which each seller $j \in J$ has initial endowment $\omega^j \in \mathbb{Z}^L_+$. We define

$$\bar{Y}^j = \{x \in \mathbb{Z}^L_+ : 0 \leq x_l \leq \omega^j_l \text{ for all } l \in L\}.$$

Suppose that each seller $j \in J$ has a utility function (or reservation price) $V^j : \bar{Y}^j \to \mathbb{R}$. We define the demand correspondence $D^j : \mathbb{R}^L_+ \rightrightarrows \bar{Y}^j$ by

$$D^j(p) = \arg\max_{x \in \bar{Y}^j} \{V^j(x) + p \cdot (\omega - x)\}.$$

For each seller $j \in J$, let us define the cost function C^j by

$$C^j(x) = V^j(\omega) - V^j(\omega - x) \text{ for all } x \in \bar{Y}^j. \tag{2}$$

Namely, $C^j(x)$ represents the decrease in the utility by giving $\omega - x$ to other agents. One can check that $S^j(p) = \{\omega - x : x \in D^j(p)\}$ for all $p \in \mathbb{R}^L_+$. In this way, we can identify seller j with utility function V^j as the seller with cost function C^j.

2.3 Core and competitive equilibrium

Define $N = H \cup J$. A *coalition* is a subset S of N such that $S \cap H \neq \emptyset$, $S \cap J \neq \emptyset$. For each coalition S, a tuple $(x^i, t^i)_{i \in S}$ is called an S-*allocation* iff

$$
\begin{aligned}
(x^h, I^h - t^h) &\in X \times \mathbb{R}_+ \text{ for all } h \in S \cap H, \\
(x^j, t^j) &\in Y^j \times \mathbb{R} \text{ for all } j \in S \cap J, \\
\sum_{h \in S \cap H} (x^h, t^h) &= \sum_{j \in S \cap J} (x^j, t^j).
\end{aligned}
$$

In words, a tuple is an S-allocation if it is attainable by reallocating commodities and money among agents in S. An N-allocation $(x^i, t^i)_{i \in N}$ is simply called an *allocation* and denoted as (x, t).

An allocation (x, t) is called *individually rational* iff

$$
\begin{aligned}
U^h(x^h, I^h - t^h) &\geq U^h(\mathbf{0}, I^h) \text{ for all } h \in H, \\
t^j - C^j(x^j) &\geq 0 \text{ for all } j \in J.
\end{aligned}
$$

Namely, each agent ends up at least the utility (or profit) level that he enjoys by his own.

We say that a coalition S can *improve upon* allocation (x, t) iff there exists an S-allocation $(y^i, u^i)_{i \in S}$ such that

$$
\begin{aligned}
U^h(y^h, I^h - u^h) &\geq U^h(x^h, I^h - t^h) \text{ for all } h \in S \cap H, \\
u^j - C^j(y^j) &\geq t^j - C^j(x^j) \text{ for all } j \in S \cap J,
\end{aligned}
$$

with strict inequality holding for at least one member in S.[1] A *core allocation* is an individually rational allocation that no coalition can improve upon. The set of all core allocations is denoted as $\mathcal{C}$.

We say that a tuple $((x^i)_{i \in N}, p)$ is a *competitive equilibrium* iff

$$
\begin{aligned}
p &\in \mathbb{R}^L_+, \\
x^h &\in D^h(p) \text{ for all } h \in H, \\
x^j &\in S^j(p) \text{ for all } j \in J, \\
\sum_{h \in H} x^h &= \sum_{j \in J} y^j.
\end{aligned}
$$

We say that an allocation (x, t) is a *competitive allocation* iff there exists $p \in \mathbb{R}^L_+$ such that

$$
\begin{aligned}
&t^i = p \cdot x^i \text{ for all } i \in N, \\
&((x^i)_{i \in N}, p) \text{ is a competitive equilibrium.}
\end{aligned}
$$

The set of all competitive allocations is denoted as $\mathcal{E}$.

[1] Under A1, it is equivalent to assuming that all inequalities hold with strict inequalities.

3 Discrete convex analysis

In this section, we explain M$^\natural$-convex function introduced in discrete convex analysis. We also refer to previous results on the relationship between M$^\natural$-concavity and the gross substitutes condition.

3.1 Definition and interpretation

For each $x \in \mathbb{Z}^L$, we define

$$\mathrm{supp}^+ x = \{l \in L : x_l > 0\},\ \mathrm{supp}^- x = \{l \in L : x_l < 0\}.$$

We say that C^j is *M$^\natural$-convex* iff for any $x, y \in Y^j$ and $l \in \mathrm{supp}^+(x-y)$, there exists $m \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that

$$C^j(y + \mathbf{1}_l - \mathbf{1}_m) - C^j(y) \leq C^j(x) - C^j(x - \mathbf{1}_l + \mathbf{1}_m). \tag{3}$$

Figure 1 below depicts the four points in (3) in the case of $|L| = 2$:

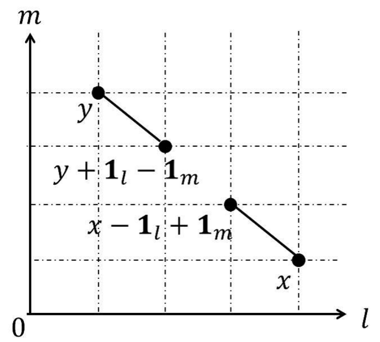

Figure 1

We explain why the inequality (3) represents convexity. Let us first revisit an ordinary convex function on continuous domain. Consider a function $f : \mathbb{R}_+ \to \mathbb{R}$. Then, f is convex iff for any three points $x, y, z \in \mathbb{R}_+$ with $x < y < z$,

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Namely, convexity is represented by comparing the slopes at different intervals.

When $|L| = 1$, the definition of M$^\natural$-convex function reduces to a discrete version of the above inequality. $C^j : \mathbb{Z}_+ \to \mathbb{R} \cup \{+\infty\}$ is M$^\natural$-convex iff for any $x \in \mathbb{Z}_+$ with $x \geq 1$,

$$C^j(x) - C^j(x-1) \leq C^j(x+1) - C^j(x). \tag{4}$$

This inequality represents *non-decreasing marginal costs.*

In (4), we compare the cost of producing one more unit. We can interpret that (3) extends this idea to functions with multi-dimensional domain. In (3), we compare the cost of producing one more unit of l, while removing one more unit of m. Let us call the cost *marginal cost of l for m.*[2] Notice that, from the assumption $l \in \text{supp}^+(x-y)$, the l-th coordinate of $x - \mathbf{1}_l + \mathbf{1}_m$ is no less than that of y. Then, (3) states that the marginal cost of l for m at $x - \mathbf{1}_l + \mathbf{1}_m$ is no less than that at y.

Consider an M$^\natural$-convex function $C^j : \mathbb{Z}_+^L \to \mathbb{R}$. We give two properties of C^j that help us understand the shape of M$^\natural$-convex function. First, it is convex in each coordinate:

$$\begin{aligned}&\text{For each } x \in \mathbb{Z}_+^L \text{ with } x_l \geq 1, \text{ we have}\\ &C^j(x) - C^j(x - \mathbf{1}_l) \leq C^j(x + \mathbf{1}_l) - C^j(x).\end{aligned}$$

Second, the 'cross derivative' is always non-negative:

$$\begin{aligned}&\text{For each } x \in \mathbb{Z}_+^L \text{ and } l, m \in L, l \neq m, \text{ we have}\\ &C^j(x + \mathbf{1}_l + \mathbf{1}_m) - C^j(x + \mathbf{1}_m) \geq C^j(x + \mathbf{1}_l) - C^j(x).\end{aligned}$$

Let us apply the two properties to 2-dimensional case, $L = \{l, m\}$. The first property states that, with the m-th coordinate being fixed, C^j gradually increases in l; see Figure 2. The second property states that the difference in l-th coordinate increases as the value of m-th coordinate increases; see Figure 3.

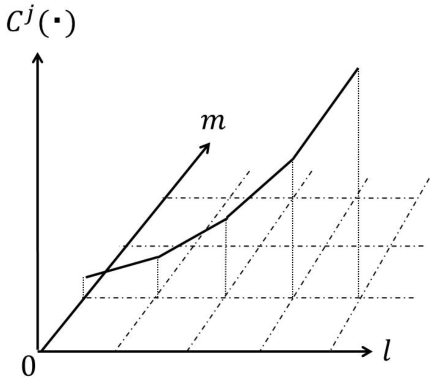

Figure 2

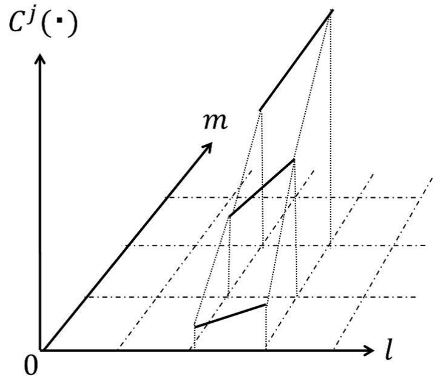

Figure 3

We make the following assumption:

A4 (*M$^\natural$-convexity*). $C^j(\cdot)$ is M$^\natural$-convex.

[2]In the case of $m = 0$, we just compare the cost of producing one more unit of l.

3.2 Gross substitute

In the literature on two-sided market, the gross-substitute (GS) condition has played a central role. Kelso-Crawford (1982) proved that, under the condition, there exists a core allocation in job matching model. The condition has been adapted to more general models; see, for example, Hatfield and Milgrom (2005) or Hatfield et al. (2013).

In this subsection, we refer to previous studies on the relationship between M$^\natural$-concave function and the GS condition. Let us go back to the market model in Subsection 2.2 where each seller j has a utility function and initial endowments. We say that V^j is M$^\natural$-concave iff $-V^j$ is M$^\natural$-convex. One can check that, if V^j is M$^\natural$-concave, then the cost function C^j defined by (2) is M$^\natural$-convex.

(GS) For any two price vectors $p, q \in \mathbb{R}^L_+$ with $p \leq q$ and $x \in D^j(p)$, there exists $y \in D^j(q)$ such that, for any $l \in L$,

$$p_l = q_l \text{ implies } y_l \geq x_l.$$

Proposition 1 (Murota and Tamura (2003)) *If V^j is M$^\natural$-concave, then V^j satisfies GS.*

The next proposition states that, when a seller j initially owns at most one unit of each commodity and V^j is monotone-nondecreasing, GS and M$^\natural$-concavity are equivalent.

Proposition 2 (Murota and Tamura (2003)) *Suppose that $\bar{Y}^j \subseteq \{0,1\}^L$ and V^j is monotone-nondecreasing. Then, V^j satisfies GS if and only if V^j is M$^\natural$-concave.*

Gul and Stachetti (1999) proved that, under the supposition of Proposition 2, GS is also equivalent to the single-improvement property.

4 Main results

4.1 Existence and coincidence

We are now in a position to state the main results. Let us introduce an additional notation. We say that two sellers j and k, $j \neq k$, are the *same type* iff $C^j(x) = C^k(x)$ for all $x \in \mathbb{Z}^L_+$.

Theorem 1 *Assume A1 to A4. If for each $j \in J$, there is at least one seller $k \in J$ who is the same type as j, then $\mathcal{C} = \mathcal{E}$.*

Proof. See Appendix A.

The set-inclusion $\mathcal{C} \supseteq \mathcal{E}$ is standard. The main message of Theorem 1 is the converse set-inclusion, $\mathcal{C} \subseteq \mathcal{E}$. Namely, if an individually rational allocation (x, t) is not improved upon by any coalition, then there exists a competitive price that supports the allocation. Moreover, as we show in the proof, we can find a competitive price in a constructive way.

Without the if-clause of the statement, we can find a counter-example where the coincidence does not hold. Intuitively, without the if-clause, the Core permits price discrimination, i.e., a seller may trade the same commodity at different prices in a core allocation. For a more detailed discussion, see Lemma 2.2 of Moulin (1995) or the discussion after Theorem 10 of Kaneko (1982).

By making an additional assumption, we can guarantee the existence of $\mathcal{E}$.

A5 (*Finiteness*). If $U^h(x, c) > U^h(\mathbf{0}, I^h)$, then $U^h(x, c) = U^h(\mathbf{0}, I^h + \delta)$ for some $\delta > 0$.

A5 states that the commodity can be compensated by money.

Theorem 2 *Assume A1 to A5. Then,* $\mathcal{E} \neq \emptyset$.

Proof. See Appendix B.

4.2 Sketch of proof of Theorem 1

We give a sketch of proof of $\mathcal{C} \subseteq \mathcal{E}$. In particular, we explain the role of M$^\natural$-convex function in the proof.

Let a core allocation (x, t) be given. We first show that, if a commodity is traded at (x, t) (i.e., if there exists a seller who produces the commodity), then the commodity is always traded at the same price; this part can be proved by using the if-clause of the statement. We also show that, for each commodity that is not traded at (x, t), there exists a price at which no buyer wants to buy, and no seller wants to sell the commodity. As a consequence, we can find a price for all commodities. Let p denote the price vector.

The remaining task is to show that, at price vector p, buyers maximize their utilities, and sellers maximize their profits. Here, we only explain profit maximization for sellers. We use the following property:[3]

[3]See Murota and Tamura (2003).

Proposition 3 (Single improvement property) *Suppose that C^j satisfies A4. If $x \in Y^j$ satisfies*

$$p \cdot x - C^j(x) \geq p \cdot (x + \mathbf{1}_l - \mathbf{1}_m) - C^j(x + \mathbf{1}_l - \mathbf{1}_m)$$
$$\textit{for all } l, m \in L \cup \{0\} \textit{ such that } x + \mathbf{1}_l - \mathbf{1}_m \in Y^j,$$

then $x \in S^j(p)$.

Proposition 3 states the following: in order to conclude that x attains the maximum profit, it suffices to check that the profit does not increase 'by either removing a commodity from x, adding a commodity to x, or doing both' (Tamura (2004)).[4]

Consider a seller j who produces x^j. Then, there exists a set of buyers H' who consume the commodities, i.e., $\sum_{h \in H'} x^h = x^j$. Suppose that $x^j_l \geq 1$; equivalently, there is a buyer $h \in H'$ who consumes commodity l. Suppose also that there is a buyer $h' \in H \backslash H'$ who consumes commodity m.

Now, consider the situation in which j produces $x^j - \mathbf{1}_l + \mathbf{1}_m$, instead of x^j. We can describe the situation in the context of coalition formation as follows: seller j stops selling one unit of l to h, and starts selling one unit of m to h'. Namely, j forms a coalition $\{j\} \cup (H' \backslash \{h\}) \cup \{h'\}$. Figure 4 below visualizes the situation in the case of $H' = \{h, i, i'\}$. Straight line between agents means that the agents trade commodities and money.

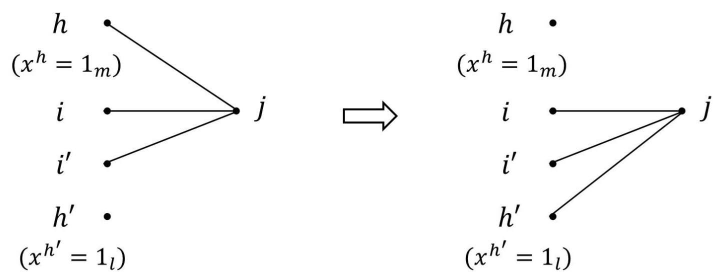

Figure 4

We can now apply core stability. Consider the following allocation for $\{j\} \cup (H' \backslash \{h\}) \cup \{h'\}$: every buyer h consumes x^h, j produces $x^j - \mathbf{1}_l + \mathbf{1}_m$, and the commodities are traded at the price p. Then, core stability requires that j's profit does not increase, i.e.,

$$p \cdot x^j - C^j(x^j) \geq p \cdot (x^j + \mathbf{1}_l - \mathbf{1}_m) - C^j(x^j + \mathbf{1}_l - \mathbf{1}_m).$$

[4]This is comparable to the following property of an ordinary convex function: local optimum implies global optimum.

This is exactly the inequality in the single improvement property. So, we can connect the single improvement property with core stability. By considering all possible deviations, we can show that j cannot increase his profit by removing or adding one unit of a commodity. From Proposition 3, profit maximization for j follows.

5 Concluding remarks

We conclude this paper by referring to two remaining problems.

The first problem is to study the structure of the core and the set of competitive prices. Previous studies have shown that, under the gross substitutes condition, the Core has a lattice structure; see, for example, Hatfield and Milgrom (2005). We conjecture that the Core and the competitive equilibria in our market model also have a lattice structure.

The second problem is to study the maximality of the domain of $M^\natural$-convex function. There are some previous results showing that a certain condition is not only sufficient for the existence of a stable allocation, but also maximal in the following sense: if some agent violates the condition, we can construct a market in which a stable allocation no longer exists. For a reference, see Theorem 2 of Gul and Stacchetti (1999) or Theorem 7 of Hatfield et al. (2013). It remains as a future task to figure out whether or not $M^\natural$-convexity yields a similar result.

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Appendix A

We prove Theorem 1. Let us introduce an additional notation. For any allocation (x,t) and $l \in L$, we define

$$J_l(x,t) = \{j \in J : x_l^j \geq 1\}.$$

In words, $J_l(x,t)$ is the set of sellers who produce the commodity l at the allocation (x,t).

In the proof, we define the summation over empty set to be the 0-vector in an appropriate domain.[5]

Proof of Theorem 1: The proof of $\mathcal{E} \subseteq \mathcal{C}$ is standard. We prove $\mathcal{C} \subseteq \mathcal{E}$. Choose an arbitrary core allocation $(x,t) \in \mathcal{C}$. From the definition of an allocation, we have

$$\sum_{h \in H} x^h = \sum_{j \in J} x^j, \tag{5}$$

$$\sum_{h \in H} t^h = \sum_{j \in J} t^j. \tag{6}$$

Let us introduce some notation. We change the set of buyers H into $H \cup \{\theta\}$ by adding a new buyer θ who satisfies the following:

$$U^\theta(x,c) = 0 \text{ for all } (x,c) \in X \times \mathbb{R}_+, \; x^\theta = \mathbf{0}, \; t^\theta = 0.$$

For each $j \in J$, we define $\mathcal{H}^j$ by

$$\mathcal{H}^j = \Big\{ H^j \subseteq H : \sum_{h \in H^j} x^h = x^j \Big\}.$$

The proof consists of two steps. In Step 1, we construct a price vector p. In Step 2, we show that (x,p) is a competitive equilibrium and (x,t) is a competitive allocation.

Step 1: Construction of a price vector p.

Lemma 1 *Let $j \in J$ and $H^j \in \mathcal{H}^j$. Then, there exist disjoint subsets $\{H^k\}_{k \in J}$ of H such that*

$$H^k \in \mathcal{H}^k \text{ for all } k \in J \backslash \{j\},$$
$$x^h = \mathbf{0} \text{ for all } h \notin \cup_{k \in J} H^k.$$

[5]For example, consider the equation $\sum_{i \in S} a^i = a^j$ and suppose that $S = \emptyset$. If a^j is a real number, this equation means that both sides are equal to 0. If a^j is a vector in $\mathbb{Z}^L$, this equation means that both sides are equal to the 0-vector.

Proof. Without loss of generality, suppose that $J = \{1, \cdots, n\}$ and let $1 = j$. For $r = 2, \cdots, n$, we inductively choose a subset $H^r \subseteq H$ such that

$$H^r \subseteq H \backslash (\cup_{q=1}^{r-1} H^q), \;\; H^r \in \mathcal{H}^r.$$

From (5), H^r exists for all $r = 2, \cdots, n$. From the definition of H^r,

$$\sum_{h \in \cup_{r=1}^{n} H^r} x^h = \sum_{k \in J} x^k.$$

Again from (5), $x^h = \mathbf{0}$ for all $h \notin \cup_{k \in J} H^k$. □

Lemma 2 *For any $j \in J$, $H^j \in \mathcal{H}^j$ implies $t^j = \sum_{h \in H^j} t^h$.*

Proof. Let $j \in J$ and $H^j \in \mathcal{H}^j$. Choose disjoint subsets $\{H^k\}_{k \in J}$ of H that satisfy the conditions in Lemma 1. Since (x, t) is a core allocation, for any $k \in J$ with $H^k \neq \emptyset$, coalition $S = H^k \cup \{k\}$ cannot improve upon (x, t) by the following S-allocation $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} y^h &= x^h, & u^h &= t^h & &\text{for all } h \in H^k, \\ y^k &= x^k, & u^k &= \sum_{h \in H^k} t^h. & & \end{aligned}$$

Thus,

$$t^k \geq \sum_{h \in H^k} t^h \text{ for all } k \in J, H^k \neq \emptyset. \tag{7}$$

From individual rationality,

$$t^k \geq 0 \text{ for all } k \in J, H^k = \emptyset. \tag{8}$$

Since $x^h = \mathbf{0}$ for all $h \notin \cup_{k \in J} H^k$, together with individual rationality and A1,

$$0 \geq t^h \text{ for all } h \notin \cup_{k \in J} H^k. \tag{9}$$

By taking the sum of (7), (8) and (9),

$$\sum_{k \in J} t^k \geq \sum_{h \in H} t^h.$$

From (6), the inequalities (7), (8) and (9) hold with equality. This establishes the desired equation. □

Lemma 3 *For any $h \in H$, $x^h = \mathbf{0}$ implies $t^h = 0$.*

Proof. Let $h \in H$ be a buyer such that $x^h = \mathbf{0}$. Let $j \in J$. Since $x^h = \mathbf{0}$, there exists $H^j \in \mathcal{H}^j$ such that $h \notin H^j$. From Lemma 2,

$$t^j = \sum_{i \in H^j} t^i.$$

Since $H^j \cup \{h\} \in \mathcal{H}^j$, again from Lemma 2,

$$t^j = \sum_{i \in H^j \cup \{h\}} t^i.$$

The above two equations establish the desired equation. □

Lemma 4 *Let $j, k \in J$, $j \neq k$, be sellers who are the same type. Let $y^j \in Y^j$ and $H^k \in \mathcal{H}^k$. Suppose that $y^j_l > x^k_l$ and there exists a buyer $h \in H \backslash H^k$ such that $x^h = \mathbf{1}_l$. Then, there exists $m \in supp^-(y^j - x^k) \cup \{0\}$ such that for any $h \in H \backslash H^k$, $x^h = \mathbf{1}_l$, and $h' \in H^k \cup \{\theta\}$, $x^{h'} = \mathbf{1}_m$,*

$$-C^j(y^j) \leq -t^h + t^{h'} - C^j(y^j - \mathbf{1}_l + \mathbf{1}_m).$$

Proof. Let $C(\cdot) := C^j(\cdot) = C^k(\cdot)$. From M$^\natural$-convexity, for $y^j, x^k \in \mathbb{Z}^L$ and l, there exists $m \in \text{supp}^-(y^j - x^k) \cup \{0\}$ such that

$$C(y^j) + C(x^k) \geq C(y^j - \mathbf{1}_l + \mathbf{1}_m) + C(x^k + \mathbf{1}_l - \mathbf{1}_m). \tag{10}$$

Choose arbitrary buyers $h \in H \backslash H^k$, $x^h = \mathbf{1}_l$, and $h' \in H^k \cup \{\theta\}$, $x^{h'} = \mathbf{1}_m$. By multiplying -1 and adding t^k to both sides of (10),

$$\begin{aligned} & - C(y^j) + t^k - C(x^k) \\ \leq & - t^h + t^{h'} - C(y^j - \mathbf{1}_l + \mathbf{1}_m) + t^k + t^h - t^{h'} - C(x^k + \mathbf{1}_l - \mathbf{1}_m). \end{aligned} \tag{11}$$

Define $S = (H^k \backslash \{h'\}) \cup \{h, k\}$. Consider the following tuple $(z^i, v^i)_{i \in S}$:

$$\begin{aligned} z^i &= x^i, & v^i &= t^i & \text{for all } i \in S \backslash \{k\}, \\ z^k &= x^k + \mathbf{1}_l - \mathbf{1}_m, & v^k &= t^k + t^h - t^{h'}. \end{aligned}$$

Since $H^k \in \mathcal{H}^k$, together with Lemma 2, $(z^i, v^i)_{i \in S}$ is an S-allocation. Since (x, t) is a core allocation,

$$t^k - C(x^k) \geq t^k + t^h - t^{h'} - C(x^k + \mathbf{1}_l - \mathbf{1}_m). \tag{12}$$

Inequalities (11) and (12) imply the desired inequality. □

Lemma 5 *Let $j, k \in J$, $j \neq k$, be sellers who are the same type. Let $H^j \in \mathcal{H}^j$, $H^k \in \mathcal{H}^k$, $H^j \cap H^k = \emptyset$. Suppose that $x_l^j > x_l^k$. Then, there exists $m \in$ supp$^-(x^j - x^k) \cup \{0\}$ such that for any $h \in H^j$, $x^h = \mathbf{1}_l$, and $h' \in H^k \cup \{\theta\}$, $x^{h'} = \mathbf{1}_m$,*

$$C^j(x^j) = C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) + t^h - t^{h'}.$$

Proof. We choose $m \in \text{supp}^-(x^j - x^k) \cup \{0\}$ that satisfies the condition in the statement of Lemma 4. Choose arbitrary buyers $h \in H^j$, $x^h = \mathbf{1}_l$, and $h' \in H^k \cup \{\theta\}$, $x^{h'} = \mathbf{1}_m$. Then, from Lemma 4,

$$t^j - C^j(x^j) \leq t^j - t^h + t^{h'} - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m). \tag{13}$$

Define $S = (H^j \backslash \{h\}) \cup \{h', j\}$. Consider the following tuple $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} &y^i = x^i, && y^i = t^i && \text{for all } i \in S \backslash \{j\}, \\ &y^j = x^j - \mathbf{1}_l + \mathbf{1}_m, && y^j = t^j - t^h + t^{h'}. \end{aligned}$$

Since $H^j \in \mathcal{H}^j$, together with Lemma 2, $(y^i, u^i)_{i \in S}$ is an S-allocation. Since (x, t) is a core allocation,

$$t^j - C^j(x^j) \geq t^j - t^h + t^{h'} - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m). \tag{14}$$

Inequalities (13) and (14) imply the desired equation. □

Lemma 6 *Let $l \in L$, $J_l(x, t) \neq \emptyset$. Then,*

$$h, h' \in H \text{ and } x^h = x^{h'} = \mathbf{1}_l \text{ imply } t^h = t^{h'}.$$

Proof. **Case 1:** Suppose that $|J_l(x, t)| = 1$. Let $\{j\} = J_l(x, t)$. Choose buyers $h, h' \in H$ who satisfy $x^h = x^{h'} = \mathbf{1}_l$. Let k be a seller who is the same type as j. Choose $H^j \in \mathcal{H}^j$, $H^k \in \mathcal{H}^k$, $H^j \cap H^k = \emptyset$. Since $\{j\} = J_l(x, t)$, $x_l^j > x_l^k$ and $h, h' \in H^j$. Thus, from Lemma 5, there exists $m \in \text{supp}^-(x^j - x^k) \cup \{0\}$ such that for any $h'' \in H^k \cup \{\theta\}$, $x^{h''} = \mathbf{1}_m$,

$$\begin{aligned} C^j(x^j) &= C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) + t^h - t^{h''}, \\ C^j(x^j) &= C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) + t^{h'} - t^{h''}. \end{aligned}$$

By taking the difference of the above two equations, we obtain the desired equation.

Case 2: Suppose that $|J_l(x, t)| \geq 2$. Choose distinct buyers $h, h' \in H$ such that $x^h = x^{h'} = \mathbf{1}_l$. Let a seller $j \in J_l(x, t)$ be given. Since $|J_l(x, t)| \geq 2$, there exists $H^j \in \mathcal{H}^j$ such that $h \in H^j$ and $h' \notin H^j$. From Lemma 2,

$$t^j = \sum_{i \in H^j} t^i. \tag{15}$$

Consider the coalition $(H^j \backslash \{h\}) \cup \{h'\}$. Since $(H^j \backslash \{h\}) \cup \{h'\} \in \mathcal{H}^j$, again from Lemma 2,

$$t^j = \sum_{i \in (H^j \backslash \{h\}) \cup \{h'\}} t^i. \tag{16}$$

From equations (15) and (16), we obtain $t^h = t^{h'}$. □

Let $l \in L$, $J_l(x,t) \neq \emptyset$. We define p_l by

$$p_l = t^h, \text{ where } h \in H, x^h = \mathbf{1}_l. \tag{17}$$

Note that, from Lemma 6, p_l does not depend on the choice of $h \in H$, $x^h = \mathbf{1}_l$. We prove that $p_l \geq 0$; choose $h \in H$ and $j \in J$ who satisfy $x^h = \mathbf{1}_l$ and $x_l^j \geq 1$. Consider the following tuple $(y^i, u^i)_{i \in N \backslash \{h\}}$:

$$\begin{aligned} y^i &= x^i, & u^i &= t^i & &\text{for all } i \in N \backslash \{h, j\}, \\ y^j &= x^j - \mathbf{1}_l, & u^j &= t^j - t^h. \end{aligned}$$

From (6), $(y^i, u^i)_{i \in N \backslash \{h\}}$ is an $N \backslash \{h\}$-allocation. Since (x,t) is a core allocation,

$$\begin{aligned} t^j - C^j(x^j) &\geq t^j - t^h - C^j(x^j - \mathbf{1}_l), \\ t^h &\geq C^j(x^j) - C^j(x^j - \mathbf{1}_l) \geq 0, \end{aligned}$$

where the last inequality holds from A3. We obtain $t^h = p_l \geq 0$.

We define $p_0 = 0$ in the remaining part. Let $m \in L$, $J_m(x,t) = \emptyset$. We define p_m^i for all $i \in N$ as follows:

- Let $j \in J$ be a seller such that $x^j - \mathbf{1}_l + \mathbf{1}_m \in Y^j$ for some $l \in L \cup \{0\}$. We define p_m^j by

  $$p_m^j = \min_{l \in L \cup \{0\}} \{p_l + C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) - C^j(x^j)\}.$$

  Note that p_l is already defined by (17).

- Let $j \in J$ be a seller such that $x^j - \mathbf{1}_l + \mathbf{1}_m \notin Y^j$ for all $l \in L \cup \{0\}$. We define $p_m^j = +\infty$.

- Let $h \in H$ be a buyer such that $U^h(x^h, I^h - t^h) > U^h(\mathbf{1}_m, I^h)$. We define $p_m^h = 0$.

- Let $h \in H$ be a buyer such that $U^h(\mathbf{1}_m, I^h) \geq U^h(x^h, I^h - t^h)$. From A2,

  $$U^h(\mathbf{0}, I^h) \geq U^h(\mathbf{1}_m, 0). \tag{18}$$

From individual rationality,

$$U^h(x^h, I^h - t^h) \geq U^h(\mathbf{0}, I^h). \tag{19}$$

From (18) and (19),

$$U^h(\mathbf{1}_m, I^h) \geq U^h(x^h, I^h - t^h) \geq U^h(\mathbf{1}_m, 0).$$

We define p_m^h as the real number that satisfies

$$U^h(x^h, I^h - t^h) = U^h\big(\mathbf{1}_m, I^h - p_m^h\big).$$

From A1, p_m^h exists and is unique.

Lemma 7 *Let $m \in L$, $J_m(x,t) = \emptyset$, and $j \in J$. Then,*

$$p_m^j \geq 0.$$

Proof. If $p_m^j = +\infty$, the result trivially holds. Suppose that $p_m^j < +\infty$. Then, there exists $l \in L \cup \{0\}$ such that

$$\begin{aligned} &p_l + C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) - C^j(x^j) = p_m^j, \\ &p_l - C^j(x^j) = p_m^j - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) \leq p_m^j - C^j(x^j - \mathbf{1}_l), \end{aligned} \tag{20}$$

where the last inequality holds from A3. If $l = 0$, the result holds from (20). Suppose that $l \in L$. Choose $H^j \in \mathcal{H}^j$ and $h \in H^j$, $x^h = \mathbf{1}_l$. Define $S = (H^j \backslash \{h\}) \cup \{j\}$. Consider the following tuple $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} &y^i = x^i, && u^i = t^i && \text{for all } i \in S\backslash\{j\}, \\ &y^j = x^j - \mathbf{1}_l, && u^j = t^j - p_l. \end{aligned}$$

Since $H^j \in \mathcal{H}^j$, together with Lemma 2, $(y^i, u^i)_{i\in S}$ is an S-allocation. Since (x,t) is a core allocation,

$$\begin{aligned} &t^j - C^j(x^j) \geq t^j - p_l - C^j(x^j - \mathbf{1}_l), \\ &p_l - C^j(x^j) \geq -C^j(x^j - \mathbf{1}_l). \end{aligned}$$

Together with (20), we obtain $p_m^j \geq 0$. □

Lemma 8 *Let $m \in L$, $J_m(x,t) = \emptyset$. Then,*

$$\max_{h \in H} p_m^h \leq \min_{j \in J} p_m^j.$$

Proof. Choose arbitrary $h \in H$ and $j \in J$. It suffices to prove that $p^h_m \leq p^j_m$. If $p^h_m = 0$, the result holds from Lemma 7. If $p^j_m = +\infty$, the result trivially holds. Hence, suppose that $p^h_m > 0$ and $p^j_m < +\infty$.

From the definition of p^h_m,

$$U^h(x^h, I^h - t^h) = U^h(\mathbf{1}_m, I^h - p^h_m). \tag{21}$$

For seller j, there exists $l \in L \cup \{0\}$ such that

$$\begin{aligned} p_l + C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) - C^j(x^j) = p^j_m, \\ t^j - C^j(x^j) = t^j - p_l + p^j_m - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m). \end{aligned} \tag{22}$$

We define $x^h = \mathbf{1}_\alpha$, $\alpha \in L \cup \{0\}$. We divide the remaining part into two cases.

Case 1: Suppose that $\{\alpha = 0\}$ or $\{\alpha \in L$ and $J_\alpha(x,t) \neq \{j\}\}$. Then, there exists $H^j \in \mathcal{H}^j$ such that $h \notin H^j$. Choose $h' \in H^j \cup \{\theta\}$, $x^{h'} = \mathbf{1}_l$, and define $S = (H^j \backslash \{h'\}) \cup \{h, j\}$. Consider the following tuple $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} &y^i = x^i, && u^i = t^i && \text{for all } i \in S\backslash\{h,j\}, \\ &y^h = \mathbf{1}_m, && u^h = p^h_m, && \\ &y^j = x^j - \mathbf{1}_l + \mathbf{1}_m, && u^j = t^j - p_l + p^h_m. && \end{aligned}$$

Since $H^j \in \mathcal{H}^j$, together with Lemma 2, $(y^i, u^i)_{i \in S}$ is an S-allocation. From (21) and the fact that (x,t) is a core allocation, we have

$$t^j - C^j(x^j) \geq t^j - p_l + p^h_m - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m).$$

From the above inequality and (22), we obtain $p^j_m \geq p^h_m$.

Case 2: Suppose that $\alpha \in L$ and $J_\alpha(x,t) = \{j\}$.

Subcase 2-1: Suppose that $\alpha = l \in L$. Then, there exists $H^j \in \mathcal{H}^j$ such that $h \in H^j$. Define $S = \{j\} \cup H^j$. Consider the following tuple $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} &y^i = x^i, && u^i = t^i && \text{for all } i \in S\backslash\{h,j\}, \\ &y^h = \mathbf{1}_m, && u^h = p^h_m, && \\ &y^j = x^j - \mathbf{1}_l + \mathbf{1}_m, && u^j = t^j - p_l + p^h_m. && \end{aligned}$$

Since $H^j \in \mathcal{H}^j$, together with Lemma 2, $(y^i, u^i)_{i \in S}$ is an S-allocation. From (21) and the fact that (x,t) is a core allocation, we have

$$t^j - C^j(x^j) \geq t^j - p_l + p^h_m - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m).$$

From the above inequality and (22), we obtain $p^j_m \geq p^h_m$.

Subcase 2-2 Suppose that $\alpha \neq l$. Let k be a seller who is the same type as j. Choose $H^j \in \mathcal{H}^j$, $H^k \in \mathcal{H}^k$, $H^j \cap H^k = \emptyset$. Since $J_\alpha = \{j\}$, $h \in H^j$. We can check that $\alpha \in \text{supp}^+(x^j - \mathbf{1}_l + \mathbf{1}_m - x^k)$ from the following observations:

- Since $J_\alpha(x,t)=\{j\}$ and $\alpha \neq l$, the α-th coordinate of $x^j - \mathbf{1}_l + \mathbf{1}_m$ is no less than 1.
- Since $k \notin J_\alpha(x,t)$, $x^k_\alpha = 0$.

From Lemma 4, there exists $\beta \in \mathrm{supp}^-(x^j - \mathbf{1}_l + \mathbf{1}_m - x^k) \cup \{0\}$ such that

$$\begin{aligned} & t^j - p_l + p^j_m - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) \\ \leq & t^j - p_l + p^j_m - p_\alpha + p_\beta - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m - \mathbf{1}_\alpha + \mathbf{1}_\beta). \end{aligned}$$

From the above inequality and (22), we obtain

$$t^j - C^j(x^j) \leq t^j - p_l + p^j_m - p_\alpha + p_\beta - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m - \mathbf{1}_\alpha + \mathbf{1}_\beta). \quad (23)$$

Choose $h' \in H^j \cup \{\theta\}$, $x^{h'} = \mathbf{1}_l$, and $h'' \in H^k \cup \{\theta\}$, $x^{h''} = \mathbf{1}_\beta$. Define $S = (H^j \backslash \{h'\}) \cup \{h'', j\}$. Consider the following tuple $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} y^i &= x^i, & u^i &= t^i & \text{for all } i \in S \backslash \{h, j\}, \\ y^h &= \mathbf{1}_m, & u^h &= p^h_m, & \\ y^j &= x^j - \mathbf{1}_l + \mathbf{1}_m - \mathbf{1}_\alpha + \mathbf{1}_\beta, & u^j &= t^j - p_l + p^h_m - p_\alpha + p_\beta. & \end{aligned}$$

Since $H^j \in \mathcal{H}^j$, together with Lemma 2, $(y^i, u^i)_{i \in S}$ is an S-allocation. From (21) and the fact that (x,t) is a core allocation,

$$t^j - C^j(x^j) \geq t^j - p_l + p^h_m - p_\alpha + p_\beta - C(x^j - \mathbf{1}_l + \mathbf{1}_m - \mathbf{1}_\alpha + \mathbf{1}_\beta).$$

From the above inequality and (23), we obtain $p^j_m \geq p^h_m$. □

For each $m \in L$, $J_m(x,t) = \emptyset$, choose a real number p_m that satisfies

$$\max_{h \in H} p^h_m \leq p_m \leq \min_{j \in J} p^j_m.$$

From Lemma 8, p_m is well-defined. From Lemma 7, $p_m \geq 0$.

We remark that, for each $m \in L$, $J_m(x,t) = \emptyset$, we have $\min_{j \in J} p^j_m < +\infty$. To see this, choose a seller $j \in J$ who satisfies $\mathbf{1}_m \in Y^j$; such a seller j always exists from the assumption (1). From M$^\natural$-convexity, for $\mathbf{1}_m, x^j \in \mathbb{Z}^L_+$, there exists $m' \in \mathrm{supp}^-(\mathbf{1}_m - x^j) \cup \{0\}$ such that

$$C^j(\mathbf{1}_m) + C^j(x^j) \geq C^j(\mathbf{1}_{m'}) + C^j(x^j - \mathbf{1}_{m'} + \mathbf{1}_m).$$

Thus, we obtain $x^j - \mathbf{1}_{m'} + \mathbf{1}_m \in Y^j$. In particular, $p^j_m < +\infty$, which implies the desired inequality.

Step 2: (x,p) is a competitive equilibrium.

Lemma 9 *Let $h \in H$ and $x \in X_p^h$, $x \neq x^h$. Then,*

$$U^h(x^h, I^h - p \cdot x^h) \geq U^h(x, I^h - p \cdot x).$$

Proof. We define $x^h = \mathbf{1}_\alpha$ and $x = \mathbf{1}_l$ for $\alpha, l \in L \cup \{0\}$, $\alpha \neq l$. If $l = 0$, individual rationality establishes the desired inequality. So, suppose that $l \in L$. We consider two cases.
Case 1: Suppose that $J_l(x,t) \neq \emptyset$.
Subcase 1-1: Suppose that $\{\alpha = 0\}$ or $\{\alpha \in L$ and $|J_\alpha(x,t) \cup J_l(x,t)| \geq 2\}$. Choose a seller $j \in J$ in the following way:

(a) If $\alpha = 0$, choose arbitrary $j \in J_l(x,t)$.

(b) If $\alpha \in L$ and $J_l(x,t) \backslash J_\alpha(x,t) \neq \emptyset$, choose $j \in J_l(x,t) \backslash J_\alpha(x,t)$.

(c) If $\alpha \in L$ and $J_l(x,t) \subseteq J_\alpha(x,t)$, choose arbitrary $j \in J_l(x,t)$.

Note that, in case (c), $|J_\alpha(x,t)| \geq 2$. Thus, in either case, there exists $H^j \in \mathcal{H}^j$ such that $h \notin H^j$. Choose $h' \in H^j$, $x^{h'} = \mathbf{1}_l$. Define $S = (H^j \backslash \{h'\}) \cup \{h, j\}$. Consider the following tuple $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} &y^i = x^i, && u^i = t^i && \text{for all } i \in S \backslash \{h\}, \\ &y^h = \mathbf{1}_l, && u^h = p_l. \end{aligned}$$

Since $H^j \in \mathcal{H}^j$, together with Lemma 2, $(y^i, u^i)_{i \in S}$ is an S-allocation. Since (x,t) is a core allocation, we have

$$U^h(x^h, I^h - p \cdot x^h) \geq U^h(\mathbf{1}_l, I^h - p_l).$$

Subcase 1-2: Suppose that $\alpha \in L$ and $|J_\alpha(x,t) \cup J_l(x,t)| = 1$. Let $\{j\} = J_\alpha(x,t) \cup J_l(x,t)$. Let k be a seller who is the same type as j. Choose $H^j \in \mathcal{H}^j$, $H^k \in \mathcal{H}^k$, $H^j \cap H^k = \emptyset$. Note that $h \in H^j$.

Since $k \notin J_\alpha(x,t)$, $x^j_\alpha > x^k_\alpha$. Thus, from Lemma 5, there exists $\beta \in \text{supp}^-(x^j - x^k) \cup \{0\}$ such that

$$t^j - C^j(x^j) = t^j - p_\alpha + p_\beta - C(x^j - \mathbf{1}_\alpha + \mathbf{1}_\beta). \tag{24}$$

Choose $h' \in H^k \cup \{\theta\}$, $x^{h'} = \mathbf{1}_\beta$, and $h'' \in H^j$, $x^{h''} = \mathbf{1}_l$. Define $S = (H^j \backslash h'') \cup \{h', j\}$. Consider the following tuple $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} &y^i = x^i, && u^i = t^i && \text{for all } i \in S \backslash \{h, j\}, \\ &y^h = \mathbf{1}_l, && u^h = p_l, \\ &y^j = x^j - \mathbf{1}_\alpha + \mathbf{1}_\beta, && u^j = t^j - p_\alpha + p_\beta. \end{aligned}$$

Since $H^j \in \mathcal{H}^j$, together with Lemma 2, $(y^i, u^i)_{i \in S}$ is an S-allocation. Since (x, t) is a core allocation, together with (24),

$$U^h(x, I^h - p \cdot x^h) = U^h(\mathbf{1}_\alpha, I^h - p_\alpha) \geq U^h(\mathbf{1}_l, I^h - p_l).$$

Case 2: Suppose that $J_l(x, t) = \emptyset$.
Subcase 2-1: Suppose that $U^h(x^h, I^h - p \cdot x^h) > U^h(\mathbf{1}_l, I^h)$. Since $U^h(\mathbf{1}_l, I^h) \geq U^h(\mathbf{1}_l, I^h - p_l)$ from A1, we obtain the result.
Subcase 2-2: Suppose that $U^h(x^h, I^h - p \cdot x^h) \leq U^h(\mathbf{1}_l, I^h)$. Then, from the definition of p_l^h and A1,

$$U^h(x^h, I^h - p \cdot x^h) = U^h\big(\mathbf{1}_l, I^h - p_l^h\big) \geq U^h\big(\mathbf{1}_l, I^h - p_l\big).$$

□

Lemma 10 *Let $j \in J$ be a seller such that $x_l^j \geq 1$. Then,*

$$p \cdot x^j - C^j(x^j) \geq p \cdot (x^j - \mathbf{1}_l) - C^j(x^j - \mathbf{1}_l).$$

Proof. Let $H^j \in \mathcal{H}^j$. Choose $h \in H^j$, $x^h = \mathbf{1}_l$. Define $S = (H^j \setminus \{h\}) \cup \{j\}$. Consider the following tuple $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} y^i &= x^i, & u^i &= t^i & &\text{for all } i \in S \setminus \{j\}, \\ y^j &= x^j - \mathbf{1}_l, & u^j &= p \cdot (x^j - \mathbf{1}_l). \end{aligned}$$

Since $H^j \in \mathcal{H}^j$, together with Lemma 2, $(y^i, u^i)_{i \in S}$ is an S-allocation. Since (x, t) is a core allocation, we have

$$p \cdot x^j - C^j(x^j) \geq p \cdot (x^j - \mathbf{1}_l) - C^j(x^j - \mathbf{1}_l).$$

□

Lemma 11 *Let $j \in J$ be a seller such that $x^j - \mathbf{1}_l + \mathbf{1}_m \in Y^j$ for some $l \in L \cup \{0\}$, $m \in L$, $l \neq m$. Then,*

$$p \cdot x^j - C^j(x^j) \geq p \cdot (x^j - \mathbf{1}_l + \mathbf{1}_m) - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m).$$

Proof. **Case 1:** Suppose that $J_m(x, t) \neq \emptyset$.
Subcase 1-1: Suppose that $J_m(x, t) \neq \{j\}$. Choose $h \in H$, $x^h = \mathbf{1}_m$. From the supposition, there exists $H^j \in \mathcal{H}^j$ such that $h \notin H^j$. Choose

$h' \in H^j \cup \{\theta\}$, $x^{h'} = \mathbf{1}_l$, and define $S = (H^j \backslash \{h'\}) \cup \{h, j\}$. Consider the following tuple $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} y^i &= x^i, & u^i &= t^i & &\text{for all } i \in S \backslash \{j\}, \\ y^j &= x^j - \mathbf{1}_l + \mathbf{1}_m, & u^j &= p \cdot (x^j - \mathbf{1}_l + \mathbf{1}_m), \end{aligned}$$

Since $H^j \in \mathcal{H}^j$, together with Lemma 2, $(y^i, u^i)_{i \in S}$ is an S-allocation. Since (x, t) is a core allocation, we have

$$p \cdot x^j - C^j(x^j) \geq p \cdot (x^j - \mathbf{1}_l + \mathbf{1}_m) - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m).$$

Subcase 1-2: Suppose that $J_m(x, t) = \{j\}$. Let k be a seller who is the same type as j. Choose $H^j \in \mathcal{H}^j$, $H^k \in \mathcal{H}^k$, $H^j \cap H^k = \emptyset$. We can check that $m \in \text{supp}^+(x^j - \mathbf{1}_l + \mathbf{1}_m - x^k)$ from the following observations:

- Since $J_m(x, t) = \{j\}$, the m-th coordinate of $x^j - \mathbf{1}_l + \mathbf{1}_m$ is no less than 1.
- Since $k \notin J_m(x, t)$, $x^k_m = 0$.

From Lemma 4, there exists $\beta \in \text{supp}^-(x^j - \mathbf{1}_l + \mathbf{1}_m - x^k) \cup \{0\}$ such that

$$\begin{aligned} -p_m + p_\beta - C^j(x^j - \mathbf{1}_l + \mathbf{1}_\beta) &\geq -C^j(x^j - \mathbf{1}_l + \mathbf{1}_m), \\ t^j - p_l + p_\beta - C^j(x^j - \mathbf{1}_l + \mathbf{1}_\beta) &\geq t^j - p_l + p_m - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m). \end{aligned} \tag{25}$$

Choose $h \in H^j \cup \{\theta\}$, $x^h = \mathbf{1}_l$, and $h' \in H^k \cup \{\theta\}$, $x^{h'} = \mathbf{1}_\beta$. Define coalition $S = (H^j \backslash \{h\}) \cup \{h', j\}$. Consider the following tuple $(y^i, u^i)_{i \in S}$:

$$\begin{aligned} y^i &= x^i, & u^i &= t^i & &\text{for all } i \in S \backslash \{j\}, \\ y^j &= x^j - \mathbf{1}_l + \mathbf{1}_\beta, & u^j &= t^j - p_l + p_\beta. \end{aligned}$$

Since $H^j \in \mathcal{H}^j$, together with Lemma 2, $(y^i, u^i)_{i \in S}$ is an S-allocation. Since (x, t) is a core allocation,

$$t^j - C(x^j) \geq t^j - p_l + p_\beta - C(x^j - \mathbf{1}_l + \mathbf{1}_\beta).$$

Together with (25), we obtain the desired inequality.
Case 2: Suppose that $J_m(x, t) = \emptyset$. From the definition of p^j_m,

$$p_m \leq p^j_m \leq p_l + C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) - C^j(x^j). \tag{26}$$

Thus, we have

$$\begin{aligned} & p \cdot (x^j - \mathbf{1}_l + \mathbf{1}_m) - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) \\ \leq\ & p \cdot (x^j - \mathbf{1}_l) + p_l + C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) - C^j(x^j) - C^j(x^j - \mathbf{1}_l + \mathbf{1}_m) \\ =\ & p \cdot x^j - C^j(x^j), \end{aligned}$$

where the inequality holds from (26). □

We resume the proof of Theorem 1. Lemma 9 implies that $x^h \in D^h(p)$ for all $h \in H$. Lemmas 10, 11 and Proposition 3 imply that $x^j \in S^j(p)$ for all $j \in J$. From the definition of p and Lemma 3, $t^h = p \cdot x^h$ for all $h \in H$. From Lemma 2, $t^j = p \cdot x^j$ for all $j \in J$. Thus, (x, t) is a competitive allocation. $\Box$

Appendix B

We prove Theorem 2. Let us give some preliminaries. For a function $f : \mathbb{Z}^L \to \mathbb{R} \cup \{\pm\infty\}$, we define

$$\text{dom } f = \{x \in \mathbb{Z}^L : -\infty < f(x) < +\infty\}.$$

A function $f : \mathbb{Z}^L \to \mathbb{R} \cup \{+\infty\}$ is M$^\natural$-convex iff dom $f \neq \emptyset$ and for any $x, y \in$ dom f and $l \in \text{supp}^+(x-y)$, there exists $m \in \text{supp}^-(x-y) \cup \{0\}$ such that

$$f(y + \mathbf{1}_l - \mathbf{1}_m) - f(y) \leq f(x) - f(x - \mathbf{1}_l + \mathbf{1}_m).$$

A function $f : \mathbb{Z}^L \to \mathbb{R} \cup \{-\infty\}$ is M$^\natural$-concave iff $-f$ is M$^\natural$-convex.

For an M$^\natural$-concave function f and $x \in$ dom f, we define the set of supergradients $\partial f(x)$ as follows:

$$\partial f(x) = \{p \in \mathbb{R}^L : f(x) + p \cdot (z - x) \geq f(z) \text{ for all } z \in \mathbb{Z}^L\}.$$

We use the following properties:[6]

Proposition 4 *Let f be an M$^\natural$-concave function. Then, $\partial f(x) \neq \emptyset$ for all $x \in$ dom f.*

Proposition 5 *Let f, g be M$^\natural$-concave functions. Consider the following function h:*

$$h(z) = \sup\{f(x) + f(y) : x, y \in \mathbb{Z}^L, x + y = z\} \textit{ for all } z \in \mathbb{Z}^L.$$

If $h(z) < +\infty$ for all $z \in \mathbb{Z}^L$, then h is also an M$^\natural$-concave function.

Danilov et al. (2001) proved the existence of a competitive equilibrium in the Arrow-Debreu model with indivisibility. In the proof,[7] they first construct an auxiliary utility function that depends on a price vector. Then, Kakutani's theorem is applied to the market in which agents have the auxiliary utility functions. We borrow this idea in the proof below.

[6]See Theorems 6.61 and 6.15 of Murota (2003).

[7]See the proof of Proposition 2 in the Appendix.

Proof of Theorem 2: Let $h \in H$. We assume that U^h is normalized by $U^h(\mathbf{0}, I^h) = 0$. With a slight abuse of notation, we extend the domain of cost function C^j from $\mathbb{Z}^L_+$ to $\mathbb{Z}^L$ by assuming that $C^j(x) = +\infty$ for all $x \notin \mathbb{Z}^L_+$.

We define

$$A^h = \{x \in X : U^h(x, I^h) \geq U^h(\mathbf{0}, I^h)\}.$$

Let $x \in A^h$ be fixed. From A2,

$$U^h(x, I^h) \geq U^h(\mathbf{0}, I^h) \geq U^h(x, 0).$$

Thus, from A1, there exists a unique real number d^h_x, $0 \leq d^h_x \leq I^h$, such that

$$U^h(x, I^h - d^h_x) = U^h(\mathbf{0}, I^h) = 0.$$

From A1, for any $p \in \mathbb{R}^L_+$, $U^h(x, I^h - p \cdot x) \geq 0$ if and only if $0 \leq p \cdot x \leq d^h_x$. From A5, there exists $w^h_p(x) \geq 0$ such that

$$U^h(x, I^h - p \cdot x) = U^h(\mathbf{0}, I^h + w^h_p(x)).$$

From A1, $w^h_p(x)$ is unique. Figure 5 below visualizes $w^h_p(x)$:

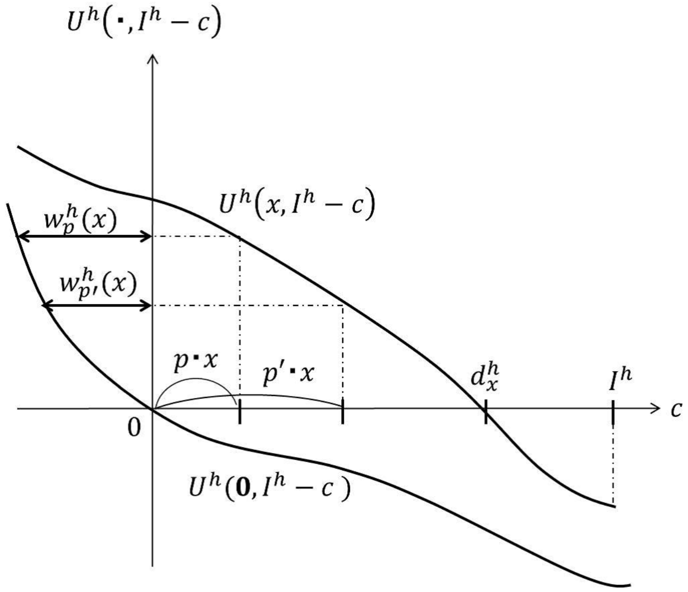

Figure 5

For each $p \in \mathbb{R}^L_+$ and $x \in X$, we define $u^h_p(x)$ by[8]

$$u^h_p(x) = \begin{cases} w^h_p(x) + p \cdot x & \text{if } x \in A^h, p \cdot x \leq d^h_x, \\ d^h_x & \text{if } x \in A^h, p \cdot x > d^h_x. \\ -1 & \text{if } x \notin A^h. \end{cases}$$

[8]We remark that, when U^h is quasi-linear, u^h_p does not depend on the choice of p.

Note that, for any $x \in X$, $u_p^h(x)$ is continuous with respect to p since for any $x \in A^h$ and $p \in \mathbb{R}_+^L$ with $p \cdot x = d_x^h$, we have $u_p^h(x) = d_x^h$.

For each $h \in H$ and $x \in X$, we define

$$B_x^h = \max\{u_p^h(x) : p \in \mathbb{R}_+^L, 0 \le p \cdot x \le d_x^h\}.$$

For each $p \in \mathbb{R}_+^L$, we define $u_p^h(x) = -\infty$ for all $x \in \mathbb{Z}^L$, $x \notin X$. Then, we obtain a function $u_p^h : \mathbb{Z}^L \to \mathbb{R} \cup \{-\infty\}$. Since dom $u_p^h = X$, u_p^h is an M$^\natural$-concave function.[9]

Lemma 12 *Let $p \in \mathbb{R}_+^L$. Then,*

$$\begin{aligned} & U^h(x^h, I^h - p \cdot x^h) \ge U^h(x, I^h - p \cdot x) && \textit{for all } x \in X_p^h \\ \Leftrightarrow\ & u_p^h(x^h) - p \cdot x^h \ge u_p^h(x) - p \cdot x && \textit{for all } x \in X. \end{aligned}$$

Proof. **Proof of $\Rightarrow$:** From the assumption,

$$U^h(x^h, I^h - p \cdot x^h) \ge U^h(x, I^h - p \cdot x) \text{ for all } x \in A^h, p \cdot x \le d_x^h.$$

Since $U^h(x^h, I^h - p \cdot x^h) \ge U^h(\mathbf{0}, I^h) = 0$, we have $x^h \in A^h$ and $p \cdot x^h \le d_{x^h}^h$. Thus, from the definition of $w_p^h(\cdot)$,

$$U^h(\mathbf{0}, I^h + w_p^h(x^h)) \ge U^h(\mathbf{0}, I^h + w_p^h(x)) \text{ for all } x \in A^h, p \cdot x \le d_x^h.$$

From A1,

$$w_p^h(x^h) \ge w_p^h(x) \text{ for all } x \in A^h, p \cdot x \le d_x^h.$$

From the definition of $u_p^h(\cdot)$,

$$u_p^h(x^h) - p \cdot x^h \ge u_p^h(x) - p \cdot x \text{ for all } x \in A^h, p \cdot x \le d_x^h. \tag{27}$$

By letting $x = \mathbf{0}$ in (27), we have $u_p^h(x^h) - p \cdot x^h \ge 0$. On the other hand,

$$u_p^h(x) - p \cdot x < 0 \text{ for all } x \in A^h, p \cdot x > d_x^h. \tag{28}$$

Note that, from the definition of u_p^h, $u_p^h(x) - p \cdot x < 0$ for all $x \notin A^h$. This fact and equations (27) and (28) establish the desired inequality.

Proof of $\Leftarrow$: From the assumption,

$$u_p^h(x^h) - p \cdot x^h \ge u_p^h(x) - p \cdot x \text{ for all } x \in A^h, p \cdot x \le d_x^h.$$

[9]One can check that the inequality in the definition of M$^\natural$-concave function always holds with equality.

Since $u_p^h(x^h) - p \cdot x^h \geq u_p^h(\mathbf{0}) = 0$, we have $x^h \in A^h$ and $p \cdot x^h \leq d_{x^h}^h$. Thus, from the definition of $u_p^h(\cdot)$,

$$w_p^h(x^h) \geq w_p^h(x) \text{ for all } x \in A^h, p \cdot x \leq d_x^h.$$

From A1,

$$U^h(\mathbf{0}, I^h + w_p^h(x^h)) \geq U^h(\mathbf{0}, I^h + w_p^h(x)) \text{ for all } x \in A^h, p \cdot x \leq d_x^h.$$

From the definition of $w_p^h(\cdot)$,

$$U^h(x^h, I^h - p \cdot x^h) \geq U^h(x, I^h - p \cdot x) \text{ for all } x \in A^h, p \cdot x \leq d_x^h. \tag{29}$$

By letting $x = \mathbf{0}$ in (29), we have $U^h(x^h, I^h - p \cdot x^h) \geq U^h(\mathbf{0}, I^h) = 0$. On the other hand, from the definition of d_x^h and A1,

$$U^h(x^h, I^h - p \cdot x^h) \geq U^h(\mathbf{0}, I^h) > U^h(x, I^h - p \cdot x) \qquad \text{for all } x \in A^h, d_x^h < p \cdot x \leq I^h. \tag{30}$$

From the definition of A^h and A1, for any $x \in X_p^h$, $x \notin A^h$,

$$U^h(x^h, I^h - p \cdot x^h) \geq U^h(\mathbf{0}, I^h) > U^h(x, I^h) \geq U^h(x, I^h - p \cdot x).$$

This fact and equations (29) and (30) establish the desired inequality. □

For each $p \in \mathbb{R}_+^L$, define $f_p : \mathbb{Z}^L \to \mathbb{R} \cup \{-\infty\}$ by

$$f_p(z) = \sup\Big\{\sum_{h \in H} u_p^h(x^h) - \sum_{j \in J} C^j(y^j) : \sum_{h \in H} x^h - \sum_{j \in J} y^j = z\Big\} \text{ for all } z \in \mathbb{Z}^L.$$

In order to show that f_p is an M$^\natural$-concave function, we consider the following function: for each $j \in J$,

$$\tilde{C}^j(x) = -C^j(-x) \text{ for all } x \in \mathbb{Z}^L.$$

One can check that $\tilde{C}^j(\cdot)$ is an M$^\natural$-concave function. Then, we can rewrite f_p as follows:

$$f_p(z) = \sup\Big\{\sum_{h \in H} u_p^h(x^h) + \sum_{j \in J} \tilde{C}^j(y^j) : \sum_{h \in H} x^h + \sum_{j \in J} y^j = z\Big\} \text{ for all } z \in \mathbb{Z}^L.$$

From Proposition 5, we obtain that f_p is an M$^\natural$-concave function.

From Proposition 4, $\partial f_p(\mathbf{0}) \neq \emptyset$. Since dom u_p^h is finite for all $h \in H$ and dom C^j is finite for all $j \in J$, we can find a tuple $\big((\bar{x}_p^h)_{h\in H}, (\bar{y}_p^j)_{j\in J}\big)$ that satisfies

$$\begin{aligned} f_p(\mathbf{0}) &= \sum_{h\in H} u_p^h(\bar{x}_p^h) - \sum_{j\in J} y^j(\bar{y}_p^j), \\ \sum_{h\in H} \bar{x}_p^h &= \sum_{j\in J} \bar{y}_p^j. \end{aligned} \tag{31}$$

Lemma 13 *Let $p \in \mathbb{R}_+^L$. Then, $p' \in \partial f_p(\mathbf{0})$ if and only if*

$$\begin{aligned} &u_p^h(\bar{x}_p^h) - p' \cdot \bar{x}_p^h \geq u_p^h(x^h) - p' \cdot x^h \text{ for all } x^h \in X, h \in H, \text{ and} \\ &p' \cdot \bar{y}_p^j - C^j(\bar{y}_p^j) \geq p' \cdot y^j - C^j(y^j) \text{ for all } y^j \in Y^j, j \in J. \end{aligned} \tag{32}$$

Proof. The following equivalences hold:

$$\begin{aligned} &p' \in \partial f_p(\mathbf{0}) \\ \Leftrightarrow &f_p(\mathbf{0}) \geq f_p(z) - p' \cdot z \text{ for all } z \in \mathbb{Z}^L, \\ \Leftrightarrow &\sum_{h\in H}\Big\{u_p^h(\bar{x}_p^h) - p' \cdot \bar{x}_p^h\Big\} + \sum_{j\in J}\Big\{p' \cdot \bar{y}_p^j - C^j(\bar{y}_p^j)\Big\} \\ \geq &\sum_{h\in H}\Big\{u_p^h(x^h) - p' \cdot x^h\Big\} + \sum_{j\in J}\Big\{p' \cdot y^j - C^j(y^j)\Big\} \\ &\text{for all } x^h \in X, h \in H, y^j \in Y^j, j \in J. \end{aligned} \tag{33}$$

Thus, our purpose is to prove (32) $\Leftrightarrow$ (33).
Proof of (32) $\Rightarrow$ (33)**:** This immediately follows by taking the sum of (32) for all $h \in H$, $j \in J$.
Proof of (33) $\Rightarrow$ (32)**:** We prove the contrapositive. Suppose that there exist a buyer $h' \in H$ and $\tilde{x}^{h'} \in X$ such that

$$u_p^{h'}(\tilde{x}^{h'}) - p' \cdot \tilde{x}^{h'} > u_p^{h'}(\bar{x}_p^{h'}) - p' \cdot \bar{x}_p^{h'}. \tag{34}$$

By adding $u_p^h(\bar{x}_p^h) - p' \cdot \bar{x}_p^h$ for all $h \in H\backslash\{h'\}$ and $\bar{y}_p^j - C^j(\bar{y}_p^j)$ for all $j \in J$ to both sides of (34), we obtain the negation of (33). The case in which there exists a seller j' and $\tilde{y}^{j'}$ violating (32) can be proved in the same way. □

Lemma 14 *Let $p \in \mathbb{R}_+^L$. Then, there exists $p' \in \partial f_p(\mathbf{0})$ such that $p' \geq \mathbf{0}$.*

Proof. Let $p' \in \partial f_p(\mathbf{0})$ and suppose that $p'_l < 0$ for some $l \in L$. We show that the vector that replaces p'_l with 0 is also a supergradient.

From Lemma 13,

$$u_p^h(\bar{x}_p^h) - p' \cdot \bar{x}_p^h \geq u_p^h(x^h) - p' \cdot x^h \text{ for all } x^h \in X, h \in H, \text{ and}$$
$$p' \cdot \bar{y}_p^j - C^j(\bar{y}_p^j) \geq p' \cdot y^j - C^j(y^j) \text{ for all } y^j \in Y^j, j \in J. \tag{35}$$

Since $p'_l < 0$, for each $j \in J$, $(\bar{y}_p^j)_l = 0$; otherwise, from A3, j can obtain a higher profit by producing $\bar{y}_p^j - \mathbf{1}_l$.[10] From (31), for each $h \in H$, $(\bar{x}_p^h)_l = 0$.

Define $\bar{p} \in \mathbb{R}^L$ by $\bar{p}_l = 0$, $\bar{p}_m = p_m$ for all $m \in L$, $m \neq l$. Then, the following inequality holds: for each $h \in H$,

$$u_p^h(\bar{x}_p^h) - \bar{p} \cdot \bar{x}_p^h \geq u_p^h(x^h) - \bar{p} \cdot x^h \text{ for all } x^h \in X. \tag{36}$$

Indeed, for each $h \in H$,

$$\begin{aligned} u_p^h(\bar{x}_p^h) - \bar{p} \cdot \bar{x}_p^h &= u_p^h(\bar{x}_p^h) - p' \cdot \bar{x}_p^h \geq u_p^h(\mathbf{1}_l) - p'_l > u_p^h(\mathbf{1}_l) - \bar{p}_l, \\ u_p^h(\bar{x}_p^h) - \bar{p} \cdot \bar{x}_p^h &= u_p^h(\bar{x}_p^h) - p' \cdot \bar{x}_p^h \geq u_p^h(x^h) - p' \cdot x = u_p^h(x^h) - \bar{p} \cdot x^h \\ &\qquad \text{for all } x^h \in X, x^h \neq \mathbf{1}_l, \end{aligned}$$

which establish (36).

Let $j \in J$. We show that $\bar{y}_p^j$ maximizes j's profit at price vector $\bar{p}$. For any $m, n \in L \cup \{0\}$, $m \neq l$, $n \neq l$, we have

$$\begin{aligned} \bar{p} \cdot \bar{y}_p^j - C^j(\bar{y}_p^j) = p' \cdot \bar{y}_p^j - C^j(\bar{y}_p^j) &\geq p' \cdot (\bar{y}_p^j + \mathbf{1}_m - \mathbf{1}_n) - C^j(\bar{y}_p^j + \mathbf{1}_m - \mathbf{1}_n) \\ &= \bar{p} \cdot (\bar{y}_p^j + \mathbf{1}_m - \mathbf{1}_n) - C^j(\bar{y}_p^j + \mathbf{1}_m - \mathbf{1}_n). \end{aligned} \tag{37}$$

For any $n \in L \cup \{0\}$, $n \neq l$, we have

$$\begin{aligned} \bar{p} \cdot \bar{y}_p^j - C^j(\bar{y}_p^j) &\geq \bar{p} \cdot (\bar{y}_p^j - \mathbf{1}_n) - C^j(\bar{y}_p^j - \mathbf{1}_n) \\ &\geq \bar{p} \cdot (\bar{y}_p^j + \mathbf{1}_l - \mathbf{1}_n) - C^j(\bar{y}_p^j + \mathbf{1}_l - \mathbf{1}_n), \end{aligned}$$

where the first inequality holds from (37) with $m = 0$ and the second inequality holds from $\bar{p}_l = 0$ and A3. The above inequalities imply

$$\bar{p} \cdot \bar{y}_p^j - C^j(\bar{y}_p^j) \geq \bar{p} \cdot (\bar{y}_p^j + \mathbf{1}_m - \mathbf{1}_n) - C^j(\bar{y}_p^j + \mathbf{1}_m - \mathbf{1}_n)$$
$$\text{for all } m, n \in L \cup \{0\} \text{ such that } \bar{y}_p^j + \mathbf{1}_m - \mathbf{1}_n \in Y^j.$$

From Proposition 3,

$$\bar{p} \cdot \bar{y}_p^j - C^j(\bar{y}^j) \geq \bar{p} \cdot y^j - C^j(y^j) \text{ for all } y^j \in Y^j, j \in J. \tag{38}$$

[10] We remark that, for any $y^j \in Y^j$ with $y_l^j \geq 1$, we have $y^j - \mathbf{1}_l \in Y^j$. To see this, consider the two vectors $\mathbf{0}, y^j \in Y^j$. Since $l \in \text{supp}^+(y^j - \mathbf{0})$, from M$^\natural$-convexity, we have $C^j(\mathbf{0}) + C^j(y^j) \geq C^j(\mathbf{1}_l) + C^j(y^j - \mathbf{1}_l)$. This inequality implies that $y^j - \mathbf{1}_l \in Y^j$.

From (36), (38) and Lemma 13,

$$\bar{p} \in \partial f_p(\mathbf{0}).$$

Thus, we have proved that if a supergradient has a coordinate with negative value, the vector that replaces the value with 0 is also a supergradient. By continuing this procedure for all negative coordinates, we obtain a non-negative supergradient. □

Define U by

$$U = \max\{B^h_x : h \in H, x \in X\}.$$

Then, for any $p \in \mathbb{R}^L_+$, $h \in H$ and $l \in L$, $u^h_p(\mathbf{1}_l) \leq U$. Define C by

$$C = \max\{C^j(\mathbf{1}_l) : l \in L, j \in J \text{ with } \mathbf{1}_l \in Y^j\}.$$

Lemma 15 *Let $p \in \mathbb{R}^L_+$, $p' \in \partial f_p(\mathbf{0})$ and $l \in L$. Then,*

$$p'_l \leq |H| \cdot U + C.$$

Proof. Since $p' \in \partial f_p(\mathbf{0})$,

$$p'_l \leq f_p(\mathbf{0}) - f_p(-\mathbf{1}_l).$$

The following inequality shows that $f_p(\mathbf{0})$ is bounded from above:

$$f_p(\mathbf{0}) \leq \sum_{h \in H} \max_{x \in X} u^h_p(x) \leq |H| \cdot U.$$

Next, we show that $f_p(-\mathbf{1}_l)$ is bounded from below. From the assumption (1), there exists a seller $k \in J$ such that $\mathbf{1}_l \in Y^k$. Thus,

$$f_p(-\mathbf{1}_l) \geq -C^k(\mathbf{1}_l) \geq -C.$$

The above inequalities yield the desired condition. □

Take the price cube

$$\mathbf{Q} = \{p \in \mathbb{R}^L_+ : 0 \leq p_l \leq |H| \cdot U + C \text{ for all } l \in L\}.$$

For $p \in \mathbf{Q}$, define $\mathbf{P}(p)$ by

$$\mathbf{P}(p) = \partial f_p(\mathbf{0}) \cap \mathbb{R}^L_+.$$

From Lemmas 14 and 15, $\emptyset \neq \mathbf{P}(p) \subseteq \mathbf{Q}$. Thus, $\mathbf{P}(\cdot)$ is a set-valued function from $\mathbf{Q}$ to $\mathbf{Q}$. Note that $\mathbf{Q}$ is convex, compact and $\mathbf{P}(p)$ is convex and compact for all $p \in \mathbf{Q}$ as the set of supergradients. We show that $\mathbf{P}(\cdot)$ is upper hemi-continuous.

Take a convergent sequence $\{p^k\}_{k=1}^{\infty} \subseteq \mathbf{Q}$ such that $p^k \to p^*$. For each k, there exist $\bar{x}^h_{p^k}$, $h \in H$, and $\bar{y}^j_{p^k}$, $j \in J$, that satisfy

$$f_{p^k}(\mathbf{0}) = \sum_{h \in H} u^h_{p^k}(\bar{x}^h_{p^k}) - \sum_{j \in J} C^j(\bar{y}^j_{p^k}).$$

Since X is finite, for each $h \in H$, there is an element $\bar{x}^h_*$ that appears infinitely in $\{\bar{x}^h_{p^k}\}_{k=1}^{\infty}$. Similarly for each $j \in J$, since Y^j is finite, there is an element $\bar{y}^j_*$ that appears infinitely in $\{\bar{y}^j_{p^k}\}_{k=1}^{\infty}$. It follows that we can choose a subsequence $\{p^{k'}\}_{k'=1}^{\infty} \subseteq \{p^k\}_{k=1}^{\infty}$ that satisfies, for each k',

$$\bar{x}^h_{p^{k'}} = \bar{x}^h_* \text{ for all } h \in H,\ \bar{y}^j_{p^{k'}} = \bar{y}^j_* \text{ for all } j \in J.$$

From continuity of $u^h_p(\cdot)$ with respect to p,

$$f_{p^*}(\mathbf{0}) = \sum_{h \in H} u^h_{p^*}(\bar{x}^h_*) - \sum_{j \in J} C^j(\bar{y}^j_*).$$

Choose a sequence $\{q^{k'}\}_{k'=1}^{\infty}$ such that $q^{k'} \in \mathbf{P}(p^{k'})$ for all k'. Then, for each k',

$$\begin{aligned} f_{p^{k'}}(\mathbf{0}) + q^{k'} \cdot z &\geq f_{p^{k'}}(z) \text{ for all } z \in \mathbb{Z}, \\ \sum_{h \in H} u^h_{p^{k'}}(\bar{x}^h_*) - \sum_{j \in J} C^j(\bar{y}^j_*) + q^{k'} \cdot z &\geq \sum_{h \in H} u^h_{p^{k'}}(x^h) - \sum_{j \in J} C^j(y^j) \\ &\qquad \text{for all } x^h \in X, y^j \in Y^j. \end{aligned} \tag{39}$$

Since $\{q^{k'}\}_{k'=1}^{\infty}$ is a bounded sequence, we can choose a convergent subsequence. Assume, for notational simplicity, that $\{q^{k'}\}_{k'=1}^{\infty}$ itself converges. Let q^* denote the limit point. Now, let $k' \to \infty$ in (39). Then, from continuity of $u^h_p(\cdot)$ with respect to p,

$$\begin{aligned} \sum_{h \in H} u^h_{p^*}(\bar{x}^h_*) - \sum_{j \in J} C^j(\bar{y}^j_*) + q^* \cdot z &\geq \sum_{h \in H} u^h_{p^*}(x^h) - \sum_{j \in J} C^j(y^j) \\ &\qquad \text{for all } x^h \in X, y^j \in Y^j, \\ f_{p^*}(\mathbf{0}) + q^* \cdot z &\geq f_{p^*}(z) \text{ for all } z \in \mathbb{Z}. \end{aligned}$$

Hence, $q^* \in \mathbf{P}(p^*)$, which proves upper hemi-continuity of $\mathbf{P}(\cdot)$.

From Kakutani's theorem, there exists p^* such that $p^* \in \mathbf{P}(p^*)$. Lemmas 12 and 13 imply that $\big((\bar{x}^h_{p^*})_{h \in H}, (\bar{y}^j_{p^*})_{j \in J}, p^*\big)$ is a competitive equilibrium. □